MOMENTS AND CUMULANTS ON IDENTITIES FOR BERNOULLI AND EULER NUMBERS

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Recent results interpret Bernoulli and Euler numbers as moments of certain random variables. When considering the moments and cumulants related to Bernoulli and Euler numbers, Faá di Bruno's formulas lead to several identities, through the Bell polynomials.

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1. INTRODUCTION

We begin with two identities:

$$Y_k\left(-\frac{B_2\cdot 1!}{2\cdot 2!},-\frac{B_4\cdot 2!}{4\cdot 4!},\ldots,-\frac{B_{2k}\cdot k!}{2k\cdot(2k)!}\right)=\frac{k!(2^{1-2k}-1)B_{2k}}{(2k)!}, \tag{1.1}$$

and

$$Y_k\left(\frac{B_2\cdot 1!}{2\cdot 2!},\frac{B_4\cdot 2!}{4\cdot 4!},\ldots,\frac{B_{2k}\cdot k!}{2k\cdot(2k)!}\right)=\frac{k!}{2^{2k}(2k+1)!}, \tag{1.2}$$

where B_k is the k-th Bernoulli numbers and Y_k is the k-th complete Bell polynomial, defined as follows.

DEFINITION 1. *The Bernoulli and Euler polynomials, denoted by $B_n(x)$ and $E_n(x)$, respectively, are defined via their exponential generating functions:*

$$\frac{ze^{zx}}{e^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!},\quad (|z|<2\pi), \tag{1.3}$$

and

$$\frac{2e^{zx}}{e^z+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!},\quad (|z|<\pi). \tag{1.4}$$

The Bernoulli and Euler numbers are $B_n=B_n(0)$ and $E_n=2^nE_n(1/2)$, respectively. (See, e.g., entries 24.2.3, 24.2.4, 24.2.8 and 24.2.9 in [8]).

The definition of Bell polynomials can be found in, e.g., [2, p. 134].

DEFINITION 2. *The partial or incomplete exponential Bell polynomial is defined by*

$Y_{n,\ell}(x_1,\ldots,x_{n-\ell+1})$

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$$:= \sum_{\substack{a_1+\cdots+a_{n-\ell+1}=\ell \\ a_1+2a_2+\cdots+(n-\ell+1)a_{n-\ell+1}=n}} \frac{n!}{a_1!\cdots a_{n-\ell+1}!}\left(\frac{x_1}{1!}\right)^{a_1}\cdots\left(\frac{x_{n-\ell+1}}{(n-\ell+1)!}\right)^{a_{n-\ell+1}},$$

and the n-th complete exponential Bell polynomial is defined by

$$\begin{aligned} Y_n(x_1,\ldots,x_n) &:= \sum_{\ell=1}^{n} Y_{n,\ell}(x_1,\ldots,x_{n-\ell+1}) \\ &= \sum_{a_1+2a_2+\cdots+na_n=n} \frac{n!}{a_1!\cdots a_n!}\left(\frac{x_1}{1!}\right)^{a_1}\cdots\left(\frac{x_n}{n!}\right)^{a_n}. \end{aligned} \tag{1.5}$$

Recall the two identities (1.1) and (1.2) at the very beginning. The first one is a special case of a result obtained by Rubinstein [9, eq. 9] (with $m=d=1$ and $s=1/2$); while the second is due to Hoffman [4, Prop. 2.4]. It is surprising that Rubinstein's work [9] is on arXiv since 2009, but we failed to find it published in any journal.

Inspired by the probabilistic methods, e.g., Adell and Lekuona [1] recently consider binomial identities through moments of random variables, we shall reveal that both (1.1) and (1.2) can be similarly derived by considering certain random variables and applying the Faá di Bruno's formulas on corresponding moment-cumulant pairs. More specifically, we shall prove the following eight identities.

PROPOSITION 3. *Let n be a positive integer such that $n>1$. Then, we have*

$$Y_n\left(0,-\frac{B_2}{2},-\frac{B_3}{3},\ldots,-\frac{B_n}{n}\right) = B_n\left(\frac{1}{2}\right), \tag{1.6}$$

$$Y_n\left(0,\frac{B_2}{2},\ldots,\frac{B_n}{n}\right) = \frac{1+(-1)^n}{2^{n+1}(n+1)} = \begin{cases} \frac{1}{2^n(n+1)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \tag{1.7}$$

$$Y_n\left(0,-6B_2,-\frac{56}{3}B_3,\ldots,\frac{2^n(1-2^n)}{n}B_n\right) = E_n,$$

$$Y_n\left(0,6B_2,\frac{56}{3}B_3,\ldots,\frac{2^n(2^n-1)}{n}B_n\right) = \frac{1+(-1)^n}{2} = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\begin{aligned} B_n &= -n\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!Y_{n,\ell}\left(B_1\left(\frac{1}{2}\right),\ldots,B_{n-\ell+1}\left(\frac{1}{2}\right)\right) \\ &= n\sum_{\ell=1}^{n}(-1)^{\ell-1}(\ell-1)!Y_{n,\ell}\left(0,\frac{1}{4\cdot 3},0,\ldots,\frac{1+(-1)^{n-\ell+1}}{2^{n-\ell+2}(n-\ell+2)}\right) \end{aligned}$$

$$= \frac{n}{2^n(1-2^n)} \sum_{\ell=1}^{n} (-1)^{\ell-1}(\ell-1)! Y_{n,\ell}(E_1, \ldots, E_{n-\ell+1})$$
$$= \frac{n}{2^n(2^n-1)} \sum_{\ell=1}^{n} (-1)^{\ell-1}(\ell-1)! Y_{n,\ell}\left(0, 1, \ldots, \frac{(-1)^{n-\ell+1}+1}{2}\right),$$

where (1.6) is equivalent to (1.1) and (1.7) is equivalent to (1.2).

In order to prove Proposition 3 by probabilistic method, we shall first review basic definition of random variables, moments, cumulants, Faá di Bruno's formulas, and the probabilistic interpretations of Bernoulli and Euler polynomials in Section 2. Then, In Section 3, we shall find four pairs of moments and cumulants, listed as Table 1, which imply all the identities of Proposition 3 via Faá di Bruno's formulas.

2. PRELIMINARIES

First of all, we recall the moments and cumulants for a random variable. (See, e.g., [7, Chpt. 3].)

Let X be an arbitrary random variable on $\mathbb{R}$, with *probability density function* $p(t)$ and *moments* m_n, namely,

$$m_n = \mathbb{E}[X^n] = \int_{\mathbb{R}} t^n p(t) \mathrm{d}t.$$

The *moment generating function* of X is the exponential generating function of m_n, denoted by

$$\mathbb{E}[e^{zX}] = \int_{\mathbb{R}} e^{zt} p(t) \mathrm{d}t = \sum_{n=0}^{\infty} m_n \frac{z^n}{n!}. \tag{2.1}$$

The *cumulants* κ_n are defined via the cumulant generating function $K(z)$, which is the natural logarithm of the moment generating function (2.1):

$$K(z) := \sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!} = \log\left(\mathbb{E}[e^{zX}]\right) = \log\left(\sum_{n=0}^{\infty} m_n \frac{z^n}{n!}\right).$$

The "logarithmic-exponential" relation between the moment generating function and the cumulant generating function allows us to apply the *Faá di Bruno's formulas* (see, e.g., [5, eq. 1]) to obtain

$$m_n = Y_n(\kappa_1, \ldots, \kappa_n) \tag{2.2}$$

and its inverse relation

$$\kappa_n = \sum_{\ell=1}^{n} (-1)^{\ell-1}(\ell-1)! Y_{n,\ell}(m_1, \ldots, m_{n-\ell+1}), \tag{2.3}$$

Note that (2.2) and (2.3) are our key formulas in our proof of Proposition 3.

Now, recall the definition of Bernoulli and Euler polynomials, in Definition 1. From (1.3) and (1.4), we see an important property that for positive integer k

$$B_{2k-1}\left(\frac{1}{2}\right) = B_{2k+1} = E_{2k-1} = 0 \quad \text{and} \quad B_1 = -\frac{1}{2}. \tag{2.4}$$

Next, we give the probabilistic interpretations of $B_n(x)$ and $E_n(x)$ as follows. Letting

$$p_B(t) := \frac{\pi}{2}\operatorname{sech}^2(\pi t) \quad \text{and} \quad p_E(t) := \operatorname{sech}(\pi t), \quad (t \in \mathbb{R})$$

we define two random variables L_B and L_E with density functions p_B and p_E, respectively. Then, with $i^2 = -1$,

$$B_n(x) = \mathbb{E}\left[\left(iL_B + x - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(it + x - \frac{1}{2}\right)^n p_B(t)\mathrm{d}t, \tag{2.5}$$

$$E_n(x) = \mathbb{E}\left[\left(iL_E + x - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(it + x - \frac{1}{2}\right)^n p_E(t)\mathrm{d}t. \tag{2.6}$$

See, e.g., [3, eq. 2.14] and [6, eq. 2.3] for the two expectations above.

Remark. For both random variables L_B and L_E, the moments are $|B_n(1/2)| = \mathbb{E}[L_B^n]$ and $|E_n(1/2)| = \mathbb{E}[L_E^n]$. Given a random variable X with moments m_n and density $p(t)$, the uniqueness of $p(t)$ with respect to m_n is of importance and is not always guaranteed. To prove this uniqueness, one sufficient condition is the general Carleman's condition, (see e.g., [10, p. 59])

$$\sum_{n=1}^{\infty} m_n^{\frac{1}{n}} = \infty. \tag{2.7}$$

Note that $|B_{2n}(1/2)| \sim 4\left(1-2^{1-2n}\right)\sqrt{\pi n}\,(n/(\pi e))^{2n}$, i.e., $|B_{2n}(1/2)|^{-\frac{1}{2n}} \sim e\pi/n$, and $(-1)^n E_{2n} \sim 8\sqrt{n/\pi}\,(4n/(\pi e))^{2n}$, from entries 24.4.27, 24.11.2 and 24.11.4 in [8]. By comparison test with harmonic series, for both L_B and L_E, (2.7) is guaranteed, implying the uniqueness of p_B and p_E.

3. PROOF OF PROPOSITION 3

In this section, we shall prove Proposition 3 by applying (2.2) and (2.3) on certain pairs of moments and cumulants, which are listed in the following table.

Moments	Cumulants
$\bar{m}_n = B_n\left(\frac{1}{2}\right)$	$\bar{\kappa}_n = \begin{cases} -B_n/n, & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$
$\tilde{m}_n = \begin{cases} \frac{1}{2^n(n+1)}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$	$\tilde{\kappa}_n = -\bar{\kappa}_n = \begin{cases} B_n/n, & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$
$m'_n = E_n$	$\kappa'_n = \begin{cases} 2^n(1-2^n)B_n/n & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$
$m''_n := \begin{cases} 1, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$	$\kappa''_n = -\kappa'_n = \begin{cases} 2^n(2^n-1)B_n/n & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$

Table 1: List of pairs of moment and cumulant

Proof of Proposition 3. First of all, we verify the four pairs of moments and cumulants in Table 1.
(i) Consider the random variable $\bar{X} := iL_B$ and we denote its moments by $\bar{m}_n$ and its cumulants by $\bar{\kappa}_n$. From (1.3), (2.1), and (2.5), we see

$$\mathbb{E}[e^{z\bar{X}}] = \mathbb{E}[e^{ziL_B}] = \sum_{n=0}^{\infty} B_n\left(\frac{1}{2}\right)\frac{z^n}{n!} = \frac{z/2}{\sinh(z/2)}, \tag{3.1}$$

i.e., $\bar{m}_n := B_n(1/2)$. Meanwhile, note the cumulant generating function

$$\bar{K}(z) := \sum_{n=1}^{\infty} \bar{\kappa}_n \frac{z^n}{n!} = \log\left(\frac{z/2}{\sinh(z/2)}\right).$$

Hoffman [4, pp. 279–280] verified that

$$-\bar{K}(z) = \log\left(\frac{\sinh(z/2)}{z/2}\right) = \sum_{n=1}^{\infty} \frac{B_{2n}z^{2n}}{2n(2n)!}, \tag{3.2}$$

which, by (2.4), implies $\bar{\kappa}_n = -B_n/n$ if $n > 1$; and $\bar{\kappa}_1 = 0$.
(ii) Define a random variable $\tilde{X}$ by its moment generating function

$$M_{\tilde{X}}(z) = \mathbb{E}[e^{z\tilde{X}}] = \frac{\sinh(z/2)}{z/2} = \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(2k+1)!}.$$

Denote the moments of $\tilde{X}$ by $\tilde{m}_n$ and cumulants by $\tilde{\kappa}_n$. We see that

$$\tilde{m}_n := \mathbb{E}[\tilde{X}] = \frac{1+(-1)^n}{2^{n+1}(n+1)} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{1}{2^{2k}(2k+1)}, & \text{if } n = 2k \text{ is even.} \end{cases}$$

Meanwhile, by (3.2), $\tilde{\kappa}_n = B_n/n$ for $n > 1$ and $\tilde{\kappa}_1 = 0$.
(iii) By replacement $z \mapsto iz$ in [2, p. 88], we have

$$\log(\cosh(z)) = \sum_{k=1}^{\infty} \frac{2^{2k-1}(2^{2k}-1)B_{2k}}{k} \cdot \frac{z^{2k}}{(2k)!}.$$

Also, recall the following two generating functions [8, entry 24.2.6]

$$\frac{1}{\cosh(z)} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}, \quad \left(|z| < \frac{\pi}{2}\right),$$

and [8, entry 4.33.2]

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Then, we have, for random variable $X' := 2iL_E$, by (2.6), its moments are $m'_n = \mathbb{E}[(X')^n] = E_n$ and cumulants are given by

$$\kappa'_n = \begin{cases} \frac{2^n(1-2^n)}{n} B_n & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} = \begin{cases} 2^n(1-2^n)B_n/n & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$$

(iv) Similarly, define a random variable X'', such that its moments are

$$m''_n := \mathbb{E}\left[(X'')^n\right] = \frac{1+(-1)^n}{2} = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

then the corresponding cumulants of X'' are $\kappa''_n = -\kappa'_n$.

Now, it is obvious that applying (2.2) to the four pairs in Table 1 yields the first four identities of Proposition 3 and (2.3) gives the last four different expressions of B_n in terms of incomplete Bell polynomials. The remaining is to identify (1.6) with (1.1) and to identify (1.7) with (1.2).

For (1.6), if n is odd, by (2.4) that $B_n(1/2) = 0$, it is a zero identity. Therefore, we can assume $n = 2k$ is even. From the definition (1.5), we see that nonzero terms on the right-hand side of (1.6) are of the form

$$\frac{(2k)!}{a_2!a_4!\cdots a_{2k}!}\left(-\frac{B_2}{2\cdot 2!}\right)^{a_2}\cdots\left(-\frac{B_{2k}}{2k\cdot(2k)!}\right)^{a_{2k}},$$

where $2k = n = 2a_2 + 4a_4 + \cdots + (2k)a_{2k}$. Let $b_j = a_{2j}$, for $j = 1, \ldots, k$ to see

$$\begin{aligned}
&Y_n\left(0, -\frac{B_2}{2}, -\frac{B_3}{3}, \ldots, -\frac{B_n}{n}\right)\\
=&\sum_{2a_2+\cdots+(2k)a_{2k}=2k} \frac{(2k)!}{a_2!a_4!\cdots a_{2k}!}\left(-\frac{B_2}{2\cdot 2!}\right)^{a_2}\cdots\left(-\frac{B_{2k}}{2k\cdot(2k)!}\right)^{a_{2k}}\\
=&\sum_{b_1+2b_2+\cdots+kb_k=k} \frac{(2k)!}{b_1!\cdots b_k!}\left(\frac{-\frac{B_2\cdot 1!}{2\cdot 2!}}{1!}\right)^{b_1}\cdots\left(\frac{-\frac{B_{2k}\cdot k!}{2k\cdot(2k)!}}{k!}\right)^{b_k}\\
=&\frac{(2k)!}{k!}Y_k\left(-\frac{B_2\cdot 1!}{2\cdot 2!}, -\frac{B_4\cdot 2!}{4\cdot 4!}, \ldots, -\frac{B_{2k}\cdot k!}{2k\cdot(2k)!}\right).
\end{aligned}$$

Finally, by entry 24.4.27 in [8], We obtain (1.1).

Similar simplification on (1.7) yields (1.2). □

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