THE DISCRETE SPECTRUM OF SCHRÖDINGER OPERATORS WITH δ-TYPE CONDITIONS ON REGULAR METRIC TREES

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Abstract. This paper deals with the spectral properties of the self-adjoint Schrödinger operators $\mathcal{L}_{\delta,Q} = -D^2 + Q$ with δ-type conditions on regular metric trees. Firstly, we prove that the operator $\mathcal{L}_{\delta,Q}$ given in this paper is self-adjoint if it is lower semibounded. Based on the orthogonal decomposition of the square integrable function space on a regular tree, we can reduce the operator $\mathcal{L}_{\delta,Q}$ into the direct sum of the self-adjoint Schrödinger operators $\mathfrak{A}_{\delta,Q,k}$ which are defined on intervals $[t_k,\infty)$. Then a necessary and sufficient condition is given for each operator $\mathfrak{A}_{\delta,Q,k}$ to have discrete spectrum. The condition is an analog of Molchanov's discreteness criteria. We prove that the condition is also a necessary and sufficient condition for $\mathcal{L}_{\delta,Q}$ to have discrete spectrum. Finally, using the theory of deficiency indices we get the necessary and sufficient condition for the self-adjoint Schrödinger operators with general boundary conditions to have discrete spectrum.

1. Introduction

A differential operator on a metric graph G is a system of differential operators on intervals with lengths given by the lengths of corresponding edges, and the system is complemented by appropriate matching conditions at inner vertices and by some boundary conditions at the boundary vertices. For the Schrödinger operators discussed in this paper the differential expression is

$$L_Q f(x) = -f''(x) + Q(x)f(x), \;\; x \in G \tag{1.1}$$

and the matching conditions at inner vertices are as follows:

$$\begin{cases} f_-(v) = f_1(v) = \cdots = f_{b(v)}(v), \\ f_1'(v) + \cdots + f_{b(v)}'(v) - f_-'(v) = \alpha_v f(v), \end{cases} \tag{1.2}$$

here α_v is a fixed real number depending on the vertex v, $b(v)$ is the number of edges emanating from v. We call these conditions as *δ-type conditions*. If α_v is 0 for all v, the condition (1.2) is the Kirchhoff conditions.

Our main goal is to investigate the spectral properties of self-adjoint Schrödinger operators with δ-type conditions (1.2) on regular metric trees, which are a special class of graphs with high symmetry and with no circle. The precise definitions of metric trees and regular trees are in Section 2.

Recently there has been an increasing interest in spectral theory of differential operators on metric trees. A review of spectral theory on metric trees is beyond the scope of this introduction, so we give only a partial list of works that are relevant to

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our work. For the regular metric trees with compact completion (the metric space theory needed is in [1, pp.139-170]), R. Carlson [2, 3] has shown that the spectra of Schrödinger operators with bounded potential is discrete. In fact this assertion also holds for general metric graphs with compact completion.

In the following, Γ denotes an arbitrary regular metric tree. If the longest distance between two points in Γ is infinite, we say that Γ has infinite height, in which situation the completion $\overline{\Gamma}$ is not compact. For the Schrödinger operators on a regular metric tree Γ with infinite height, in [4] the authors have estimated the total number of negative eigenvalues and the moments of these eigenvalues in terms of integrals of the symmetric (that is, depending only on the distance from x to the root) potential V.

We divide the following works of the spectral problems on a regular metric tree Γ with infinite height into three cases depending on the edge lengths of Γ, that is, the lengths of the edges in the tree Γ.

Case 1: The edge lengths of Γ are unbounded, i.e., $\sup_{e\in\mathcal{E}(\Gamma)}|e| = \infty$.

For general metric graphs, M. Solomyak has proven that when a graph G satisfies $\sup_{e\in\mathcal{E}(G)}|e| = \infty$, in which $\mathcal{E}(G)$ means the set of edges in G, the spectrum of the Laplacian on the graph G is $[0,\infty)$ [5], and this situation includes case 1.

Case 2: The edge lengths of Γ are bounded and bounded below by a positive constant S, i.e., $\sup_{e\in\mathcal{E}(\Gamma)}|e| < \infty$ and $\inf_{e\in\mathcal{E}(\Gamma)}|e| = S > 0$.

Case 3: The edge lengths of Γ are bounded and without positive lower bound, i.e., $\sup_{e\in\mathcal{E}(\Gamma)}|e| < \infty$ and $\inf_{e\in\mathcal{E}(\Gamma)}|e| = 0$.

In case 2 and case 3, there must be infinitely many edges and infinitely many vertices in Γ. In these two cases, M. Solomyak has studied the Laplacian in [6] and obtained the necessary and sufficient conditions about the branching function g_Γ (the definition will be given in Section 2) for the Laplacian to have discrete spectrum. However, as we shall see at the end of Section 5, the conditions given by M. Solomyak actually only holds for the transient trees (satisfying the condition $\int_0^\infty \frac{\mathrm{d}t}{g_\Gamma(t)} < \infty$) in case 3. Then the results given by M. Solomyak in [6] implies that for regular metric trees in case 2, the spectrum of the Laplacian couldn't be discrete. For Schrödinger operators which could be seen as the perturbed operators of the Laplacian, the spectral properties may be different from that of the Laplacian. In this paper, we give a necessary and sufficient condition independent of the branching function g_Γ for the Schrödinger operators on regular metric trees in case 2 to have discrete spectrum no matter the tree is transient or not. It is entirely different from the results given by M. Solomyak.

In [7], the homogeneous metric trees, in which all the edges have equal length and all vertices have the same number of edges emanating from them, are considered. These trees constitute a special subclass of the regular trees in case 2. It is shown that the spectrum of a Schrödinger operator on homogeneous metric trees with an even periodic potential has the band-gap structure. Moreover, in each gap, there is no more than one eigenvalue (counted without multiplicity). A. V. Sobolev and M. Solomyak also studied the spectral properties of the Schrödinger operator with potential gV on homogeneous trees in [8], in which V is a real-valued symmetric function and $g \geqslant 0$. Depending on the sign and decay of V, they gave a detailed asymptotic analysis of the counting function of the discrete eigenvalues in the limit $g \to \infty$.

The main objective of this paper is to show that the classical Molchanov's discreteness criterion (see [9]) can be extended to the case of Schrödinger operator $\mathcal{L}_{\delta,Q}$ on regular metric trees in case 2.

The methods are as follows. Firstly, we reduce the Schrödinger operator $\mathcal{L}_{\delta,Q}$ defined on the tree Γ to the direct sum of the Schrödinger operators $\mathfrak{A}_{\delta,Q,k}$ defined on intervals $[t_k,\infty)$. This reduction is based upon the basic decomposition of $L^2(\Gamma)$ for the case of regular trees [3, 6, 10]. Then we turn to investigate the spectral properties of the Schrödinger operators $\mathfrak{A}_{\delta,Q,k}$. Following from the compact embedding theorems, we obtain a necessary and sufficient condition for $\mathfrak{A}_{\delta,Q,k}$ to have discrete spectrum. To do this we use the methods given by S. Albeverio, A. Kostenko and M. Malamud in [11] and some results given by J. Yan and G. Shi in [12]. Moreover, we prove that the spectrum of the Schrödinger operator $\mathcal{L}_{\delta,Q}$ is discrete if and only if the following two conditions are satisfied: (i) The spectrum of $\mathfrak{A}_{\delta,Q,0}$ is discrete. (ii) $\min\sigma(\mathfrak{A}_{\delta,Q,k})\to\infty$, as $k\to\infty$. Finally, we find that the condition for $\mathfrak{A}_{\delta,Q,0}$ to have discrete spectrum we obtained is also a necessary and sufficient condition for $\mathcal{L}_{\delta,Q}$ to have discrete spectrum.

This paper is organized as follows. In Section 2, we introduce some necessary definitions of trees and the basic decomposition of $L^2(\Gamma)$. Section 3 contains the proof of self-adjointness of the Schrödinger operator $\mathcal{L}_{\delta,Q}$ with δ-type conditions and Dirichlet boundary conditions, and the reduction of the Schrödinger operator $\mathcal{L}_{\delta,Q}$ to the direct sum of the self-adjoint Schrödinger operators $\mathfrak{A}_{\delta,Q,k}$. In Section 4, the associated quadratic forms of $\mathfrak{A}_{\delta,Q,k}$ are given, which are of major importance for our main results. Necessary and sufficient conditions for the spectra of the operators $\mathfrak{A}_{\delta,Q,k}$ and $\mathcal{L}_{\delta,Q}$ to be discrete are given in Section 5. This section contains our main results and the discrete criteria for the self-adjoint Schrödinger operators with more general boundary conditions. At the end of this section we will illustrate that if the edges of Γ have a uniform lower bound, the discreteness conditions given by M. Solomyak can not be satisfied.

2. The Regular Metric Tree and the Basic Decomposition of $L^2(\Gamma)$

In this section we would like to recall some basic definitions about trees and the basic decomposition of the function space $L^2(\Gamma)$. We refer to [3], [6], [10] for details.

2.1. **Geometry of a Regular Tree.**

We use [6] as a general reference on trees. In order to have a well defined first derivative, the graph is directed, i.e., each edge in the graph is directed.

If two edges of a graph are incident to the same pair of vertices, then these two edges are called *parallel edges*. If a path starts at a vertex v and terminates at the same vertex v, this path is called a *cycle* in the directed graph. A *tree* is a locally finite connected graph without cycles and parallel edges. Then in a tree, the path starting at an arbitrary point x and terminating at the other point y exists and is unique, it is denoted by $\langle x,y\rangle$. In a tree the vertex o with no edge terminating at it is the *root* of the tree. The *branching number* $b(v)$ of a vertex v is defined as the number of edges emanating from v.

Definition 1. *A tree Γ' is said to be a metric tree (sometimes the notion of a weighted tree is used instead) if each edge e is assigned a positive length $|e|\in(0,\infty)$.*

Then each edge e of a metric tree can be viewed as an interval of the same length with e. Lebesgue measure on intervals extends from the edges to Γ' in the obvious way. The *distance* $\rho(x,y)$ between any two points x, y in a metric tree is defined as the lengths of the unique path joining x and y, and thus the metric topology on a tree is introduced in a natural way. For a point $x \in \Gamma'$, $|x|$ stands for the distance $\rho(x, o)$.

Let Γ' be a metric tree with a unique root o, countable vertex set $\mathcal{V}(\Gamma')$ and countable edge set $\mathcal{E}(\Gamma')$, in addition, for each vertex $v \in \mathcal{V}(\Gamma') \setminus \{o\}$ there exists a unique edge terminating at v. We also assume that $b(v) < \infty$ for any $v \in \mathcal{V}(\Gamma')$.

Adding the assumption that Γ' is a tree with its edge lengths bounded below by a positive constant S, a subtree $E \subset \Gamma'$ is compact if and only if E is closed and has only a finite number of edges.

We write $x \prec y$ if $x \in \langle o, y\rangle$ and $x \neq y$, $x \preceq y$ if $x \in \langle o, y\rangle$. For e_v^j, the i-th edge emanating from v, $1 \leqslant i \leqslant b(v)$, we write $x \succeq e_v^j$ or $e_v^j \preceq x$, if $e_v^j \subset \langle o, x\rangle$. For any vertex v, its *generation* $\operatorname{gen}(v)$ is defined as

$$\operatorname{gen}(v) = \#\{x \in \mathcal{V}(\Gamma) : x \prec v\},$$

which counts the number of vertices $x \in \mathcal{V}(\Gamma)$ satisfy the condition $x \prec v$. In another word, the generation of a vertex v is k if there are $k+1$ vertices on the unique path between o and v including the endpoints. For any edge e emanating from vertex v we define the generation of e as $\operatorname{gen}(e) = \operatorname{gen}(v)$. The only vertex such that $\operatorname{gen}(v) = 0$ is the root o. If an edge e_0 satisfies $\operatorname{gen}(e_0) = 0$, the edge e_0 emanates from the root o. We should note is that due to that the δ-type conditions (1.2) at the vertices except o is considered in this paper, a vertex v_0 could not be understood as a inner point of a certain edge even if $b(v_0) = 1$ for $v_0 \neq o$.

Definition 2. *We call a tree Γ with a unique root as a regular tree (sometimes the notion of a radial tree is used instead) if the branching number and edge lengths are functions of the distance in the tree from the root vertex.*

Or we could say that a tree Γ is a regular tree if the branching number $b(v)$ and the length $|e|$ are only depend on the generation of v and e respectively. So for a regular metric tree, $b(v_i) = b(v_j)$ and $|v_i| = |v_j|$ if $\operatorname{gen}(v_i) = \operatorname{gen}(v_j)$, then we define $b_{\operatorname{gen}(v)}$ and $t_{\operatorname{gen}(v)}$ as

$$b_{\operatorname{gen}(v)} = b(v), t_{\operatorname{gen}(v)} = |v| \tag{2.1}$$

$\operatorname{gen}(v) \in \mathbb{N}_0$, where $\mathbb{N} = \{1, 2 \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A regular tree is fully determined by two number sequences $\{b_n\}$ and $\{t_n\}$. It is clear that $t_0 = 0$ and the sequence $\{t_n\}$ is strictly increasing.

We endow the δ-type conditions (1.2) with symmetry by assuming $\alpha_{v_i} = \alpha_{v_j}$ if $\operatorname{gen}(v_i) = \operatorname{gen}(v_j)$. Then in the following sections, the δ-type conditions (1.2) are

$$\begin{cases} f_-(v) = f_1(v) = \cdots = f_{b(v)}(v), \\ f_1'(v) + \cdots + f_{b(v)}'(v) - f_-'(v) = \alpha_{\operatorname{gen}(v)} f(v). \end{cases} \tag{2.2}$$

Here we denote the only edge which terminates at a vertex $v \neq o$ as e_v^-, and the edges emanating from $v \in \mathcal{V}(\Gamma)$ as e_v^1, e_v^2,…,$e_v^{b(v)}$ for a given v. The derivative $f_j'(v)$ is computed along the edge e_v^j, and the derivative $f_-'(v)$ is computed along the edge e_v^-.

We give the meanings of some symbols we would use in what follows. We denote *the height* of Γ as h_Γ,

$$h_\Gamma = \lim_{n\to\infty} t_n = \sup_{x\in\Gamma} |x|\,.$$

In this paper, Γ has infinite height, i.e. $h_\Gamma = \infty$, then $t_n \uparrow +\infty$.

For a given vertex $v \in \mathcal{V}(\Gamma)$ and a given edge e_v^j, T_v and $T_{e_v^j}$ denote two special subtrees of Γ, which can be described as follows:

$$T_v = \{x \in \Gamma : x \succeq v\}, \quad T_{e_v^j} = e_v^j \cup \{x \in \Gamma : x \succeq e_v^j\}.$$

We have that for each $v \in \mathcal{V}(\Gamma)$

$$T_v = \cup_{1 \leqslant j \leqslant b(v)} T_{e_v^j}.$$

For $T \subset \Gamma$ is a subtree with root o_T, define *the branching function* of Γ as

$$g_T(t) = \#\{x \in T : |x| = t\}.$$

Along with the function g_T, define the functions g_k for $k \in \mathbb{N}$ as

$$g_k(t) = \#\{x \in T_e : |x| = t\}, \, \forall e \in \mathcal{E}(\Gamma) \text{ satisfying } \operatorname{gen}(e) = k.$$

It is clear that

$$g_k(t) = \begin{cases} 0, & t < t_k, \\ 1, & t_k \leqslant t \leqslant t_{k+1}, \\ b_{k+1} \dots b_n, & t_n < t \leqslant t_{n+1}, n > k, \end{cases}$$

$g_0 = (b_0)^{-1} g_\Gamma$ and $g_k(t) = (b_0 \cdots b_k)^{-1} g_\Gamma(t)$ for $t \in [t_k, h_\Gamma)$, $k \in \mathbb{N}$.

2.2. **The Basic Decomposition of $L^2(\Gamma)$.** Number the countable edges in the set $\mathcal{E}(\Gamma)$, then for the i-th directed edge e_i, we identify it with an interval $[a_i, b_i]$ of length $|e_i|$. This facilitates the discussion of function spaces and differential operators. The space $L^2(\Gamma)$ is defined as the Hilbert space $\oplus_{e_i \in \mathcal{E}(\Gamma)} L^2(e_i)$ with the inner product

$$(f, g) = \int_\Gamma f(x)\overline{g(x)}\mathrm{d}x = \sum_i \int_{a_i}^{b_i} f_i(x)\overline{g_i(x)}\mathrm{d}x,$$

where f_i, g_i are the components of f and g on the edge e_i. The inner product in $L^2(\Gamma)$ is independent of the order of edges. M. Solomyak and R. Carlson have given the orthogonal decomposition of the space $L^2(\Gamma)$ respectively in [6] and [3] in the case when Γ is a regular tree. Our further analyses are based on this decomposition.

Given a subtree $T \subset \Gamma$ with root o_T, we say that a function $f \in L^2(\Gamma)$ belongs to the set $\mathcal{F}_T$ if and only if

$$f(x) = 0 \text{ for } x \notin T; \; f(x) = f(y) \text{ if } x, y \in T \text{ and } |x| = |y|.$$

Infact, the set $\mathcal{F}_T$ is a closed subspace. When $\operatorname{gen}(e_v^j) = \operatorname{gen}(v) = k \geqslant 0$, any function $f \in \mathcal{F}_{T_{e_v^j}}$ can be naturally identified with a unique function ψ on $[t_k, h_\Gamma)$, such that $f(x) = \psi(|x|)$ for each $x \in T_{e_v^j}$ and $f(x) = 0$ outside $T_{e_v^j}$. Since $h_\Gamma = \infty$, we have

$$\int_\Gamma |f(x)|^2 \, \mathrm{d}x = \|\psi\|^2_{L^2([t_k,\infty);g_k)} := \int_{t_k}^\infty |\psi(t)|^2 \, g_k(t)\mathrm{d}t$$

for $f \in \mathcal{F}_{T_{e_v^j}}$, and

$$\int_\Gamma |f'(x)|^2 \, \mathrm{d}x = \|\psi'\|^2_{L^2([t_k,\infty);g_k)} := \int_{t_k}^\infty |\psi'(t)|^2 \, g_k(t)\mathrm{d}t$$

for $f \in \mathcal{F}_{T_{e_v^j}} \cap W^{1,2}(\Gamma)$, where $W^{1,2}(\Gamma)$ is the space consisting of all continuous functions $f \in L^2(\Gamma)$ such that $f_i \in W^{1,2}(e_i)$ for each edge $e_i \in \mathcal{E}(\Gamma)$ and

$$\|f\|^2_{W^{1,2}(\Gamma)} := \sum_{e_i \in \mathcal{E}(\Gamma)} \|f_i\|^2_{W^{1,2}(e_i)} = \int_\Gamma |f(x)|^2 \,\mathrm{d}x + \int_\Gamma |f'(x)|^2 \,\mathrm{d}x < \infty.$$

Next, we introduce a collection of subspaces $\mathcal{F}_v^{\langle s\rangle}$ of $L^2(\Gamma)$, defined for $s = 1, \ldots, b_k$ if $v = o$, and defined for $s = 1, \ldots, b_k - 1$ if $v \neq o$. For the given v, we begin with the functions $\tilde{f} \in \mathcal{F}_{T_{e_v^{b_k}}}$. The subspaces $\mathcal{F}_v^{\langle s\rangle}$ are the sets of functions satisfying

$$f(x) = \begin{cases} e^{(2\pi i s\cdot j)/b_k} \tilde{f}(y), & \text{for } x \in T_{e_v^j} : |x| = |y|, y \in T_{e_v^{b_k}}, \\ 0, & \text{for } x \notin T_v. \end{cases}$$

In the case $v = o$, the subspace $\mathcal{F}_o^{\langle b_0\rangle}$ is the function space $\mathcal{F}_\Gamma$.

The high symmetry of regular trees allows one to construct the orthogonal decomposition of the space $L^2(\Gamma)$ in Lemma 1. We call this decomposition the basic decomposition of $L^2(\Gamma)$. The following result is introduced in [3], [5], [6], and [10].

Lemma 1. *The distinct subspaces $\mathcal{F}_v^{\langle s\rangle}$ are mutually orthogonal. Moreover,*

$$L^2(\Gamma) = \mathcal{F}_\Gamma \oplus \sum_{k=0}^{\infty} \sum_{\mathrm{gen}(v)=k} \sum_{s=1}^{b_k-1} \oplus \mathcal{F}_v^{\langle s\rangle} \tag{2.3}$$

and the decomposition reduces the Laplacian on Γ.

Proof. See [3]. □

K. Naimark and M. Solomyak have described the construction of the basic decomposition of $L^2(\Gamma)$ in detail in [10]. Here we employ their description of the orthogonal projection of $f \in L^2(\Gamma)$ onto $\mathcal{F}_v^{\langle s\rangle}$.

Every function $f \in L^2(\Gamma)$ is finite almost everywhere on Γ. For a given subtree T with root o_T, a function $f \in \mathcal{F}_T$ can naturally be identified with the corresponding function $\psi \in L^2(|o_T|, h_\Gamma)$ such that $f(x) = \psi(|x|)$ almost everywhere on T. We denote the mapping as $\psi = J_T f$. The operator P_T defined as

$$(P_T f)(x) = \begin{cases} g_T(|x|)^{-1} \sum_{y \in T : |y| = |x|} f(y), & \text{for } x \in T, \\ 0, & \text{for } x \notin T, \end{cases}$$

acts on $L^2(\Gamma)$ and defines a projection onto $\mathcal{F}_T$. For a given function $f \in L^2(\Gamma)$ and a given vertex $v \in \mathcal{V}(\Gamma)$ satisfying $\mathrm{gen}(v) = k$, we define the functions $\psi_{v,f}$ and $\psi_{v,f}^j$ as

$$\psi_{v,f}(t) = (b_k g_k(t))^{-1} \sum_{y \in T_v : |y| = t} f(y) \text{ almost everywhere on } [t_k, h_\Gamma),$$

$$\psi_{v,f}^j(t) = g_k(t)^{-1} \sum_{y \in T_{e_v^j} : |y| = t} f(y) \text{ almost everywhere on } [t_k, h_\Gamma).$$

These mapping are denoted as $\psi_{v,f} = J_{T_v} P_{T_v} f$ and $\psi_{v,f}^j = J_{T_{e_v^j}} P_{T_{e_v^j}} f$. Define the vectors $\mathbf{h}_v^{\langle s\rangle}$ as

$$\mathbf{h}_v^{\langle s\rangle} = b_k^{-1/2} \left\{ e^{(2\pi i s)/b_k}, e^{(2\pi i s\cdot 2)/b_k}, \ldots, e^{(2\pi i s\cdot(b_k-1))/b_k}, 1 \right\}, \; s = 1, \ldots, b_k.$$

Then we define the function $\psi_{v,f}^{\langle s\rangle}$ as

$$\psi_{v,f}^{\langle s\rangle} = b_k^{-1/2} \sum_{j=1}^{b_k} e^{-(2\pi i s j)/b_k} \psi_{v,f}^{j}, \tag{2.4}$$

and define the vector-valued function $\boldsymbol{\psi}_{v,f}^{\langle s\rangle}$ as

$$\boldsymbol{\psi}_{v,f}^{\langle s\rangle} = \mathbf{h}_v^{\langle s\rangle} \psi_{v,f}^{\langle s\rangle} = \mathbf{h}_v^{\langle s\rangle} (b_k^{-1/2} \sum_{j=1}^{b_k} e^{-(2\pi i s j)/b_k} \psi_{v,f}^{j}), \tag{2.5}$$

where $s = 1, \dots, b_k$ if $k = 0$, and $s = 1, \dots, b_k - 1$ if $k > 0$.

If a function g on Γ belongs to the function space $\mathcal{F}_{T_{e_v^1}} \oplus \cdots \oplus \mathcal{F}_{T_{e_v^{b_k}}}$, we can define a vector-valued function $\tilde{J}_v g \in (L^2[t_k, h_\Gamma))^{b_k}$ given by

$$\tilde{J}_v g = \{g^1, \dots g^{b_k}\},\ g^i = \psi_{v,g}^i.$$

It is easy to see that the mapping $\tilde{J}_v : \mathcal{F}_{T_{e_v^1}} \oplus \cdots \oplus \mathcal{F}_{T_{e_v^{b_k}}} \to (L^2[t_k, h_\Gamma))^{b_k}$ is one-to-one for any given $v \in \mathcal{V}(\Gamma)$. The orthogonal projection from $L^2(\Gamma)$ to $\mathcal{F}_v^{\langle s\rangle}$ is given by

$$P_v^{\langle s\rangle} f = \tilde{J}_v^{-1} \boldsymbol{\psi}_{v,f}^{\langle s\rangle}.$$

And for any $v \in \mathcal{V}(\Gamma)$ the mapping

$$J_v^{\langle s\rangle} : f \in \mathcal{F}_v^{\langle s\rangle} \longmapsto \psi_{v,f}^{\langle s\rangle} \in L^2[t_k, h_\Gamma) \tag{2.6}$$

is an isometry. By the Theorem 2.3 in [10], for any function $f \in L^2(\Gamma)$ we have

$$\int_\Gamma |f(x)|^2 \, \mathrm{d}x = \int_0^{h_\Gamma} |\psi_{o,f}|^2 g_\Gamma \mathrm{d}t + \sum_{k=0}^{\infty} \sum_{\mathrm{gen}(v)=k} \sum_{s=1}^{b_k - 1} \int_{t_k}^{\infty} \left|\psi_{v,f}^{\langle s\rangle}\right|^2 g_k \mathrm{d}t.$$

3. The Schrödinger Operators

We study the differential operators in $L^2(\Gamma)$ that induced by the differential form L_Q with the potential Q. Here we employ the potential conditions given by M. Solomyak in [5]. We assume that Q is real-valued, Lebesgue measurable and symmetric on regular metric tree Γ, that means that the function value $Q(x)$ is depending on $|x|$. We can write the function Q as $Q(x) = q(t)$ for $|x| = t$. Instead of assuming that Q is bounded, we need $q \in L^1_{loc}[0, \infty)$. We define the *minimal operator* $\mathcal{L}_{\min}$ induced by L_Q as

$$D(\mathcal{L}_{\min}) = D_{\min} \text{ and } \mathcal{L}_{\min} f = L_Q f \text{ for } f \in D_{\min}.$$

The domain $D_{\min}$ is the linear span of C^∞ functions compactly supported in the interior of a single edge e_i (identified with an interval (a_i, b_i)). Correspondingly, the set $D_{\max}$ contains functions $f \in L^2(\Gamma)$ with f_i, f_i' absolutely continuous on the interval $[a_i, b_i]$ for each edge e_i and $-f'' + Qf \in L^2(\Gamma)$. The *maximal operator* $\mathcal{L}_{\max}$ induced by L_Q is defined as

$$D(\mathcal{L}_{\max}) = D_{\max} \text{ and } \mathcal{L}_{\max} f = L_Q f \text{ for } f \in D_{\max}.$$

In this paper we consider the δ-type conditions 2.2 at inner vertices $v \neq o$. One can recognize these conditions as analogues of conditions obtained from Schrödinger operators on the line with the δ potential $\sum_{k=1}^{\infty} \alpha_k \delta(t - t_k)$. If all the real number

$\alpha_{\mathrm{gen}(v)}$ in (2.2) are 0, then the δ-type conditions become the Kirchhoff conditions coming from the theory of electric networks.

We restrict our considerations to the operator $\mathcal{L}^0_{\delta,Q}$ induced by the formal operator L_Q and the domain of $\mathcal{L}^0_{\delta,Q}$ is as follows:

$$\begin{aligned}\mathrm{Dom}(\mathcal{L}^0_{\delta,Q}) = \{ & f \in L^2_{\mathrm{comp}}(\Gamma) : f(o) = 0,\ f \in D_{\max} \text{ and } f \text{ satisfies} \\ & \text{the } \delta\text{-type conditions (2.2) at the inner vertices}\},\end{aligned} \tag{3.1}$$

where $L^2_{\mathrm{comp}}(\Gamma)$ is constituted by functions in $L^2(\Gamma)$ that vanish almost everywhere outside a compact subtree.

3.1. **The Self-adjointness.** It is clear that $\mathcal{L}^0_{\delta,Q}$ is a symmetric operator. Let $\mathcal{L}_{\delta,Q}$ denote the closure of $\mathcal{L}^0_{\delta,Q}$. If $\mathcal{L}_{\delta,Q}$ is lower semibounded, then it is self-adjoint. To prove this statement, we need to find the formal operator of $(\mathcal{L}^0_{\delta,Q})^*$ firstly.

By working on one edge e_i, and using the classical theory in [14] and [15, pp.169-171], we may obtain the following result.

Lemma 2. *A function f is in the domain of the operator $(\mathcal{L}_{\min})^*$, then f belongs to $D_{\max}$ and*

$$(\mathcal{L}_{\min})^* f = L_Q f.$$

Proof. A differential operator acts componentwise on functions f in its domain. Choose an arbitrary edge e_i identified with the interval $[a_i, b_i]$, the operator $\mathcal{L}^i_{\min}$ denotes the component part operator of $\mathcal{L}_{\min}$ with domain $C_0^\infty(a_i, b_i)$, and $\mathcal{L}^i_{\max}$ is the adjoint operator of $\mathcal{L}^i_{\min}$ in $L^2(e_i)$. For each $f \in \mathrm{Dom}((\mathcal{L}_{\min})^*) \subset L^2(\Gamma)$,

$$(\mathcal{L}_{\min} g, f) = (g, (\mathcal{L}_{\min})^* f)$$

holds for all $g \in D_{\min}$. Then for each i,

$$(\mathcal{L}^i_{\min} g_i, f_i) = (g_i, (\mathcal{L}^i_{\min})^* f_i)$$

holds for all $g_i \in C_0^\infty(a_n, b_n)$, hence the function f_i satisfies the following conditions: $f_i \in L^2(e_n)$ with f_i, f_i' absolutely continuous on edge e_i and $L_Q f_i \in L^2(e_i)$, $(\mathcal{L}^i_{\min})^* f_i = \mathcal{L}^i_{\max} f_i = L_Q f_i$. That means $f \in D_{\max}$ and

$$((\mathcal{L}_{\min})^*)^i f_i = -f_i'' + Q_i f_i.$$

□

Theorem 1. *If $\mathcal{L}_{\delta,Q}$ is lower semibounded, then it is self-adjoint, $\mathcal{L}_{\delta,Q} = \mathcal{L}^*_{\delta,Q}$.*

Proof. Firstly, we prove that $\mathrm{Dom}((\mathcal{L}^0_{\delta,Q})^*)$ coincides with the set

$$\begin{aligned} D \quad = \quad & \{ f \in L^2(\Gamma) : f(o) = 0,\ f \in D_{\max} \text{ and } f \text{ satisfies} \\ & \text{the } \delta\text{-type conditions (2.2) at the inner vertices}\},\end{aligned}$$

which is a little bit different from $\mathrm{Dom}(\mathcal{L}^0_{\delta,Q})$. Since $\mathcal{L}_{\min} \subset \mathcal{L}^0_{\delta,Q}$, we have that $(\mathcal{L}^0_{\delta,Q})^* \subset (\mathcal{L}_{\min})^*$. Hence the formal operator of $(\mathcal{L}^0_{\delta,Q})^*$ is L_Q. Let $\mathcal{L}$ denote the operator with $\mathrm{Dom}(\mathcal{L}) = D$ and

$$\mathcal{L} f = L_Q f \text{ for } f \in D.$$

Integration by parts shows that $\mathcal{L}^0_{\delta,Q}$ and $\mathcal{L}$ are formal adjoints of each other. That implies $\mathcal{L} \subset (\mathcal{L}^0_{\delta,Q})^*$, it remains to prove that $(\mathcal{L}^0_{\delta,Q})^* \subset \mathcal{L}$. Let $f \in \mathrm{Dom}((\mathcal{L}^0_{\delta,Q})^*)$, then the equality

$$(\mathcal{L}^0_{\delta,Q} g, f) = (g, (\mathcal{L}^0_{\delta,Q})^* f)$$

must holds for all $g \in \mathrm{Dom}(\mathcal{L}^0_{\delta,Q})$. By Thorem 3.1 and Corollary 3.2 in [13], we get that $f \in D_{\max}$ satisfies the δ-type conditions (2.2) at the inner vertices. That implies $f \in D$. Since $\mathcal{L}^0_{\delta,Q}$ is a closable symmetric operator and $\mathcal{L}_{\delta,Q} = \overline{\mathcal{L}^0_{\delta,Q}}$, we have

$$\mathrm{Dom}((\mathcal{L}^0_{\delta,Q})^*) = \mathrm{Dom}(\mathcal{L}^*_{\delta,Q}) = D.$$

Without loss of generality, we assume that $\mathcal{L}_{\delta,Q} \geqslant I$. It is sufficient to show that $\ker(\mathcal{L}^*_{\delta,Q}) = \{0\}$, that is, the equation

$$-f''(x) + Q(x)f(x) = 0,\ x \in \Gamma\backslash\mathcal{V},\ f \in \mathrm{Dom}(\mathcal{L}^*_{\delta,Q}) \tag{3.2}$$

has only a trivial solution (derivative is understood in a distribution sense).

Recall that S is the lower bound of the edge lengths. Let $\xi \in C_0^\infty[0, S/2)$ such that $\xi(0) = 1$. Next we define a sequence of symmetric functions χ_n on Γ. Assume that $|e_1| \geqslant 1$, define the function χ_1 on Γ as

$$\chi_1(x) = \begin{cases} 1, & 0 \leqslant |x| < 1/2, \\ \xi(|x| - 1/2), & 1/2 \leqslant |x| < 1/2 + S/2, \\ 0, & |x| \geqslant 1/2 + S/2. \end{cases}$$

If $|e_1| < 1$, the function χ_1 could be defined in the same way with χ_n defined as follows. For the given Γ and $n \in \mathbb{N}$, the point $x \in \Gamma$ satisfying $|x| = n/2$ belongs to an interval $(t_k, t_{k+1}]$, where k relies on n. The choice of χ_n relies on the locations of $n/2$ and t_k, $k \in \mathbb{N}$.

Case 1: If $n/2 \in ((t_k + t_{k+1})/2, t_{k+1}]$,

$$\chi_n(x) := \begin{cases} 1, & 0 \leqslant |x| < n/2 - S/2, \\ \xi(|x| - n/2 + S/2), & n/2 - S/2 \leqslant |x| < n/2, \\ 0, & |x| \geqslant n/2. \end{cases}$$

Case 2: If $n/2 \in (t_k, (t_k + t_{k+1})/2]$,

$$\chi_n(x) := \begin{cases} 1, & 0 \leqslant |x| < n/2, \\ \xi(|x| - n/2), & n/2 \leqslant |x| < n/2 + S/2, \\ 0, & |x| \geqslant n/2 + S/2. \end{cases}$$

It is easy to see that for each v, there exists a neighbourhood O of v such that $\chi_n(v) \equiv 1$ or $\chi_n(v) \equiv 0$ in O.

Assume that $f \neq 0$ is a solution of the equation (3.2). Since f satisfies the δ-type conditions (2.2), for each vertex $v \neq o$,

$$\begin{cases} (f\chi_n)_-(v) = (f\chi_n)_1(v) = \cdots = (f\chi_n)_{b(v)}(v), \\ (f\chi_n)'_1(v) + \cdots + (f\chi_n)'_{b(v)}(v) - (f\chi_n)'_-(v) = \alpha_{\mathrm{gen}(v)} f(v)\chi_n(v), \end{cases}$$

hence $f\chi_n \in \mathrm{Dom}(\mathcal{L}^0_{\delta,Q})$. In addition $\mathcal{L}_{\delta,Q} \geqslant I$, then

$$\begin{aligned} (\mathcal{L}^0_{\delta,Q}(f\chi_n), (f\chi_n)) &= \int_\Gamma [-(f(x)\chi_n(x))'' + Q(x)f(x)\chi_n(x)]\overline{f(x)\chi_n(x)}\mathrm{d}x \\ &= -\int_\Gamma [2f'(x)\chi_n'(x) + f(x)\chi_n''(x)]\overline{f(x)\chi_n(x)}\mathrm{d}x \\ &\geqslant ((f\chi_n), (f\chi_n)) = \int_\Gamma f^2(x)\chi_n^2(x)\mathrm{d}x. \end{aligned} \tag{3.3}$$

Integrating by parts on every edge and noting that for every $v \in \mathcal{V}(\Gamma)$,

$$\chi_n'(v-) = (\chi_n)_1'(v) = \cdots = (\chi_n)'_{b(v)}(v) = 0,$$

we get

$$\begin{aligned}\int_\Gamma 2f'(x)\chi_n'(x)\overline{f(x)\chi_n(x)}\mathrm{d}x &= \frac{1}{2}\int_\Gamma (f^2(x))'(\chi_n^2(x))'\mathrm{d}x \\ &= -\int_\Gamma f^2(x)[\chi_n''(x)\chi_n(x)+(\chi_n'(x))^2]\mathrm{d}x. \qquad (3.4)\end{aligned}$$

Combining (3.3) with (3.4), we obtain

$$(\mathcal{L}^0_{\delta,Q}(f\chi_n),(f\chi_n)) = \int_\Gamma f^2(x)(\chi_n'(x))^2\mathrm{d}x.$$

Therefore, we get

$$\begin{aligned}\int_{\Gamma_{n/2-S/2}} f^2(x)\mathrm{d}x &\leqslant \int_\Gamma f^2(x)\chi_n^2(x)\mathrm{d}x \leqslant \int_\Gamma f^2(x)(\chi_n'(x))^2\mathrm{d}x \\ &\leqslant c^2(\int_{\Gamma_{[n/2+S/2]+1}} f^2(x)\mathrm{d}x - \int_{\Gamma_{[n/2-S/2]}} f^2(x)\mathrm{d}x),\end{aligned}$$

where $c := \sup_{|x|\leqslant S/2}|\xi'(t)|$, and for $m\in\mathbb{R}$, Γ_m is a subtree of Γ containing all $x\in\Gamma$, $|x|\leqslant m$. Since $f\in L^2(\Gamma)$, $f=0$. This completes the proof. □

Next, we reduce the Schrödinger operators $\mathcal{L}^0_{\delta,Q}$ and $\mathcal{L}_{\delta,Q}$. The parts of $\mathcal{L}_{\delta,Q}$ in the components of the decomposition (2.3) can be described in terms of auxiliary differential operators $\mathfrak{A}_{\delta,Q,k}$, $k\in\mathbb{N}_0$, acting in the spaces $L^2([t_k,\infty);g_\Gamma)$. A result similar with the following lemmas can be found in [5]. The relationship $A\sim B$ means that operators A and B are unitarily equivalent, and $A^{[m]}$ stands for the direct sum of m copies of a self-adjoint operator A.

Denote $L^2_{\mathrm{comp}}([t_k,\infty),g_\Gamma)$ as the set of functions in $L^2([t_k,\infty),g_\Gamma)$ with compact support. Due to that every function $f\in\mathrm{Dom}(\mathcal{L}^0_{\delta,Q})$ satisfies the boundary condition $f(o)=0$, the operator $\mathcal{L}^0_{\delta,Q}$ on Γ splits into the direct sum of operators on the subtree $T_{e_o^j}$, $j=1,\dots,b(o)$. For this reason, in the following two lemmas, and in Section 4 and Section 5 we assume $b(o)=b_0=1$.

Lemma 3. *Let Γ be a regular metric tree and Q be a real, measurable function on Γ, $Q(x)=q(|x|)$ for $x\in\Gamma$ and $q\in L^1_{loc}[0,\infty)$. The operator $\mathcal{L}^0_{\delta,Q}$ is unitarily equivalent to the direct sum of the operators $\mathfrak{A}^0_{\delta,Q,k}$:*

$$\mathcal{L}^0_{\delta,Q}\sim\mathfrak{A}^0_{\delta,Q,0}\oplus\sum_{k=1}^{\infty}\oplus(\mathfrak{A}^0_{\delta,Q,k})^{[b_1\dots b_{k-1}(b_k-1)]}. \qquad (3.5)$$

The operator $\mathfrak{A}^0_{\delta,Q,k}$ has domain

$$\begin{aligned}\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k}) = \{&\varphi\in L^2_{comp}([t_k,\infty),g_\Gamma):\varphi(t_k)=0,\ \varphi,\varphi'\in AC[t_{i-1},t_i],\\ &-\varphi''+q\varphi\in L^2([t_k,\infty),g_\Gamma),\ \varphi(t_i+)=\varphi(t_i-),\\ &b_i\varphi'(t_i+)-\varphi'(t_i-)=\alpha_i\varphi(t_i),\ \text{for all } i>k\}, \qquad (3.6)\end{aligned}$$

and

$$\mathfrak{A}^0_{\delta,Q,k}\varphi=-\varphi''+q\varphi,\ \text{for}\ \varphi\in\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k}). \qquad (3.7)$$

If the operator $\mathfrak{A}_{\delta,Q,k} := \overline{\mathfrak{A}^0_{\delta,Q,k}}$ is lower semibounded, $\mathfrak{A}_{\delta,Q,k}$ is self-adjoint, for $k\in\mathbb{N}_0$.

Proof. For a vertex $v_0 \neq o$ with $|v_0| = t_k$ and a subspace $\mathcal{F}_{v_0}^{\langle s\rangle}$ defined in Section 2, it is sufficient to show that the mapping $J_{v_0}^{\langle s\rangle}$ (see (2.6)) sends the set $\mathrm{Dom}(\mathcal{L}_{\delta,Q}^0) \cap \mathcal{F}_{v_0}^{\langle s\rangle}$ in to $L^2_{\mathrm{comp}}([t_k,\infty), g_k)$. The mapping

$$K : L^2([t_k,\infty), g_k) \to L^2([t_k,\infty), g_\Gamma)$$

defined as $Kf = (b_0 \cdots b_k)^{-1} f$, is an isometry. Denote the set $\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ as

$$\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k}) = (KJ_{v_0}^{\langle s\rangle})(\mathrm{Dom}(\mathcal{L}^0_{\delta,Q}) \cap \mathcal{F}_{v_0}^{\langle s\rangle})$$

and denote

$$\mathfrak{A}^0_{\delta,Q,k}\varphi = -\varphi'' + q\varphi, \text{ for } \varphi \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k}).$$

Next, we prove all the functions $\varphi \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ have the properties listed in (3.6). If $f \in L^2(\Gamma)$ is continuous on Γ, by (2.4) for any $v \in \mathcal{V}(\Gamma)$, we have

$$\psi^1_{v,f}(v) = \psi^2_{v,f}(v) = \cdots = \psi^{b(v)}_{v,f}(v),\ \psi_v^{\langle s\rangle}(v) = 0,\ s = 1, 2, \ldots, b_k.$$

Then every $\varphi \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ satisfies $\varphi(t_k) = 0$. Since every $f \in \mathrm{Dom}(\mathcal{L}^0_{\delta,Q}) \cap \mathcal{F}_{v_0}^{\langle s\rangle}$ satisfies the δ-type conditions (2.2) at the inner vertices, we could obtain that all the functions $\varphi \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ are continuous in the interval $[t_k,\infty)$ and satisfy the condition

$$b_i(\varphi')(t_i+) - (\varphi')(t_i-) = \alpha_i \varphi(t_i),$$

for all $i > k$. Other conditions appear in (3.6) could be obtained from the condition $\mathrm{Dom}(\mathcal{L}^0_{\delta,Q}) \subset D_{\max}$. Because of the fact that the mappings K and $J_{v_0}^{\langle s\rangle}$ are bijections, the equality (3.6) holds.

It is easy to see that

$$\mathcal{L}^0_{\delta,Q}(\mathrm{Dom}(\mathcal{L}^0_{\delta,Q}) \cap \mathcal{F}_{v_0}^{\langle s\rangle}) \subset \mathcal{F}_{v_0}^{\langle s\rangle},$$

hence $(KJ_{v_0}^{\langle s\rangle})^{-1}(\mathrm{Dom}(\mathfrak{A}_{\delta,Q,k})) \subset \mathcal{F}_{v_0}^{\langle s\rangle}$. It follows from (3.8) that (3.5) holds.

Let the strictly increasing sequence $\{t_i\}$ be defined by (2.1). For each k, the interval $[t_k,\infty)$ is a special regular tree with vertex set $\mathcal{V} = \{t_i, i \geqslant k\}$. The essential self-adjointness of operators $\mathfrak{A}^0_{\delta,Q,k}$, $k \in \mathbb{N}_0$, can be proved by the same method of Theorem 1. □

Lemma 4. *Let Γ be a regular metric tree and Q be a real, measurable and function on Γ, $Q(x) = q(|x|)$ for $x \in \Gamma$ and $q \in L^1_{loc}[0,\infty)$. The operator $\mathcal{L}_{\delta,Q}$ is unitarily equivalent to the direct sum of the operators $\mathfrak{A}_{\delta,Q,k}$:*

$$\mathcal{L}_{\delta,Q} \sim \mathfrak{A}_{\delta,Q,0} \oplus \sum_{k=1}^{\infty} \oplus \mathfrak{A}^{[b_1 \ldots b_{k-1}(b_k-1)]}_{\delta,Q,k}. \tag{3.8}$$

Proof. The proof is similar with that of Lemma 3. □

4. Quadratic Forms

We recall some basic definitions and lemmas about quadratic forms which can be found in [15]. Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let $\mathbf{t}$ be a densely defined quadratic form in $\mathfrak{H}$ with lower semibounded $-c$, that is $\mathbf{t}[u] \geqslant -c\,\|u\|^2_{\mathfrak{H}}$, $c \in \mathbb{R}$. Let $\mathbf{t}[\cdot,\cdot]$ be the sesquilinear form associated with $\mathbf{t}$ via the polarization identity. Then the equality

$$(f,g)_{\mathbf{t}} = \mathbf{t}[f,g] + (1+c)(f,g)$$

defines a scalar product on $\mathrm{Dom}(\mathbf{t})$ such that $\|u\|_{\mathbf{t}} \geqslant \|u\|_{\mathfrak{H}}$ for all $u \in \mathrm{Dom}(\mathbf{t})$, where

$$\|u\|_{\mathbf{t}}^2 := \mathbf{t}[u] + (1+c)\|u\|_{\mathfrak{H}}^2, u \in \mathrm{Dom}(\mathbf{t}).$$

The form $\mathbf{t}$ is called closable if the norms $\|\cdot\|_{\mathbf{t}}$ is compatible with $\|\cdot\|_{\mathfrak{H}}$, i.e., for every $\|\cdot\|_{\mathbf{t}}$-Cauchy sequence $\{u_n\}_{n=1}^{\infty}$ from $\mathrm{Dom}(\mathbf{t})$, $\|u_n\|_{\mathfrak{H}} \to 0$ implies $\|u_n\|_{\mathbf{t}} \to 0$. Let $\mathfrak{H}_{\mathbf{t}}$ be a $\|\cdot\|_{\mathbf{t}}$-completion of $\mathrm{Dom}(\mathbf{t})$. In this case the completion $\mathfrak{H}_{\mathbf{t}}$ can be considered as a subspace of $\mathfrak{H}$. The form $\mathbf{t}$ is called closed if the sets $\mathfrak{H}_{\mathbf{t}}$ and $\mathrm{Dom}(\mathbf{t})$ are equal.

Let A be a self-adjoint lower semibounded operator in $\mathfrak{H}$, $(Af,f) \geqslant -c(f,f)$ for all $f \in \mathrm{Dom}(A)$ and some $c \in \mathbb{R}$. Denote by $\mathbf{t}'_A$ a densely defined quadratic form, given by

$$\mathbf{t}'_A[f] = (Af,f),\ \mathrm{Dom}(\mathbf{t}'_A) = \mathrm{Dom}(A).$$

Clearly, this form is closable and lower semibounded, $\mathbf{t}'_A \geqslant -c$ and its closure $\mathbf{t}_A$ satisfies $\mathbf{t}_A \geqslant -c$. We set $\mathfrak{H}_A := \mathfrak{H}_{\mathbf{t}_A}$. By the first representation theorem (Theorem 6.2.1 in [15]), to any closed lower semibounded quadratic form $\mathbf{t} \geqslant -c$ in $\mathfrak{H}$ there corresponds a unique self-adjoint operator $A = A^*$ in $\mathfrak{H}$ satisfying $(Af,f) \geqslant -c(f,f)$ for all $f \in \mathrm{Dom}(A)$, such that $\mathbf{t}$ is the closure of $\mathbf{t}'_A$. It is uniquely determined by the conditions $\mathrm{Dom}(A) \subset \mathrm{Dom}(\mathbf{t})$ and

$$(Au,v) = \mathbf{t}[u,v],\ u \in \mathrm{Dom}(A),\ v \in \mathrm{Dom}(\mathbf{t}).$$

Lemma 5. *Let $A = A^*$ be a lower semibounded operator in $\mathfrak{H}$ and let $\mathbf{t}_A$ be the corresponding form. The spectrum $\sigma(A)$ of the operator A is discrete if and only if the embedding $i_A\colon \mathfrak{H}_A \hookrightarrow \mathfrak{H}$ is compact.*

Proof. See [15]. □

Definition 3. *Let the operator A be self-adjoint and positive in $\mathfrak{H}$ and let $\mathbf{t}_A$ be the corresponding form. The form $\mathbf{t}$ is called relatively form bounded with respect to $\mathbf{t}_A$ ($\mathbf{t}_A$-bounded) if $\mathrm{Dom}(\mathbf{t}_A) \subset \mathrm{Dom}(\mathbf{t})$ and there are positive constants a, b such that*

$$|\mathbf{t}[f]| \leqslant a\mathbf{t}_A[f] + b\|f\|_{\mathfrak{H}}^2,\ f \in \mathrm{Dom}(\mathbf{t}_A).$$

The infimum of all possible a is called the form bound of $\mathbf{t}$ with respect to $\mathbf{t}_A$. If a can be chosen arbitrary small, then $\mathbf{t}$ is called infinitesimally form bounded with respect to $\mathbf{t}_A$.

Lemma 6 (the KLMN theorem)**.** *Let $\mathbf{t}_A$ be the form corresponding to the operator $A = A^* > 0$ in $\mathfrak{H}$. If the form $\mathbf{t}$ is $\mathbf{t}_A$-bounded with relative bound $a < 1$, then the form*

$$\mathbf{t}_1 := \mathbf{t}_A + \mathbf{t},\ \mathrm{Dom}(\mathbf{t}_1) = \mathrm{Dom}\ (\mathbf{t}_A),$$

is closed and lower semibounded in $\mathfrak{H}$ and hence gives rise to a self-adjoint semibounded operator. Moreover, the norm $\|\cdot\|_A$ and $\|\cdot\|_{\mathbf{t}_1}$ are equivalent.

Proof. See [17]. □

For the rest of this section, we concentrate on the operators $\mathfrak{A}_{\delta,Q,k} = \overline{\mathfrak{A}^0_{\delta,Q,k}}$, $k \in \mathbb{N}_0$ (see (3.7)), and their corresponding quadratic forms. We start from the Hilbert space $\mathfrak{H}_k := L^2([t_k,\infty); g_\Gamma)$ and some quadratic forms in it. The quadratic forms

$$\mathfrak{a}_k[\varphi] := \int_{t_k}^{\infty} |\varphi'|^2 g_\Gamma \mathrm{d}t,\ \varphi \in \mathrm{Dom}(\mathfrak{a}_k),$$

$$\mathfrak{q}_k[\varphi] := \int_{t_k}^{\infty} q|\varphi|^2 g_\Gamma \mathrm{d}t,\ \varphi \in \mathrm{Dom}(\mathfrak{q}_k),$$

$$\mathfrak{a}_{q,k}[\varphi] := \mathfrak{a}_k[\varphi] + \mathfrak{q}_k[\varphi],\ \varphi \in \mathrm{Dom}(\mathfrak{a}_{q,k}),$$

and

$$\mathfrak{a}_{\delta,k}[\varphi] = \sum_{i=k}^{\infty} \alpha_i \left|\varphi(t_i)\right|^2 g_\Gamma(t_i),\ \varphi \in \mathrm{Dom}(\mathfrak{a}_{\delta,k})$$

are defined respectively on the domains

$$\mathrm{Dom}(\mathfrak{a}_k) = W_0^{1,2}([t_k,\infty); g_\Gamma),$$
$$\mathrm{Dom}(\mathfrak{q}_k) = \{\varphi \in L^2([t_k,\infty); g_\Gamma) : |\mathfrak{q}_k[\varphi]| < \infty\},$$
$$\mathrm{Dom}(\mathfrak{a}_{q,k}) = \{\varphi \in W_0^{1,2}([t_k,\infty); g_\Gamma) : \mathfrak{a}_{q,k}[\varphi] < \infty\},$$

and

$$\mathrm{Dom}(\mathfrak{a}_{\delta,k}) = \{\varphi \in W_0^{1,2}([t_k,\infty); g_\Gamma) : \mathfrak{a}_{\delta,k}[\varphi] < \infty\}.$$

Here $W_0^{1,2}([t_k,\infty); g_\Gamma)$ stands for the weighted Sobolev space which consists of the functions φ satisfying the following conditions: function φ and its distributional derivative φ' belong to $L^2([t_k,\infty); g_\Gamma)$, and $\varphi(t_k) = 0$.

Then we define the quadratic form $\mathfrak{a}_{\delta,q,k}$ as follows:

$$\mathfrak{a}_{\delta,q,k}[\varphi] = \mathfrak{a}_{q,k}[\varphi] + \mathfrak{a}_{\delta,k}[\varphi],\ \mathrm{Dom}(\mathfrak{a}_{\delta,q,k}) = \mathrm{Dom}(\mathfrak{a}_{q,k}) \cap \mathrm{Dom}(\mathfrak{a}_{\delta,k}).$$

If $q(t) \geqslant 0$ and $\{\alpha_i\}_{i=1}^{\infty} \subset [0,\infty)$, these quadratic forms are non-negative and closed in $\mathfrak{H}_k$, for $k \in \mathbb{N}_0$. For a given k, let $\mathcal{A}_{\delta,q,k}$ be the corresponding self-adjoint operator of $\mathfrak{a}_{\delta,q,k}$ and we find that $\mathcal{A}_{\delta,q,k}$ coincides with $\mathfrak{A}_{\delta,Q,k}$.

Lemma 7. *If $q(t) \geqslant 0$ and $\{\alpha_i\}_{i=1}^{\infty} \subset [0,\infty)$, then the form $\mathfrak{a}_{\delta,q,k}$ is non-negative and closed for each $k \in \mathbb{N}_0$.*

Proof. Let us equip $\mathfrak{H}_{\delta,q,k} = \mathrm{Dom}(\mathfrak{a}_{\delta,q,k})$ with the norm

$$\|\varphi\|^2_{\mathfrak{H}_{\delta,q,k}} = \mathfrak{a}_{q,k}[\varphi] + \mathfrak{a}_{\delta,k}[\varphi] + \|\varphi\|^2_{\mathfrak{H}_k}.$$

Let $\{\varphi_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathfrak{H}_{\delta,q,k}$. Since $W_0^{1,2}([t_k,\infty); g_\Gamma)$ and $l^2(\{\alpha_i\})$ are Hilbert spaces, there exists $\varphi \in W_0^{1,2}([t_k,\infty); g_\Gamma)$ and

$$y = \{y_i\}_{i=1}^{\infty} \in l^2(\{\alpha_i\}_{i=1}^{\infty})$$

such that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{W_0^{1,2}([t_k,\infty);g_\Gamma)} = 0$$

and

$$\lim_{n\to\infty} \sum_i \alpha_i \left|\varphi_n(t_i) - y_i\right|^2 = 0.$$

Since $g_\Gamma \geqslant 1$, $W_0^{1,2}([t_k,\infty); g_\Gamma) \subset W_0^{1,2}[t_k,\infty)$. Then the space $W_0^{1,2}([t_k,\infty); g_\Gamma)$ is continuously embedded into $C_b[t_k,\infty)$, which denotes the Banach space of bounded continuous functions on $[t_k,\infty)$. Therefore

$$\lim_{n\to\infty} \varphi_n(t_i) = \varphi(t_i),$$

and hence $y_i = \varphi(t_i)$, for all $i \geqslant k$. Then $\varphi \in \mathfrak{H}_{\delta,q,k}$ and

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{\mathfrak{H}_{\delta,q,k}} = 0.$$

In addition that $q(t) \geqslant 0$ and $\{\alpha_i\}_{i=1}^{\infty} \subset [0,\infty)$, thus $\mathfrak{H}_{\delta,q,k}$ is a Hilbert space with the inner product

$$(\varphi,\psi)_{\mathfrak{H}_{\delta,q,k}} = \int_0^{\infty} \varphi'\overline{\psi'} g_\Gamma + \int_0^{\infty} (q+1)\varphi\overline{\psi} g_\Gamma + \sum_{i=1}^{\infty} \alpha_i \varphi(t_i)\overline{\psi}(t_i) g_\Gamma(t_i).$$

Then the form $\mathfrak{a}_{\delta,q,k}$ is closed. It is obvious that the form $\mathfrak{a}_{\delta,q,k}$ is non-negative if $q(t) \geqslant 0$ and $\{\alpha_i\}_{i=1}^{\infty} \subset [0,\infty)$. □

Lemma 8. *If*

$$C_0 := \sup_{k\in\mathbb{N}} \int_{t_k}^{t_{k+1}} |q(t)|\,\mathrm{d}t < \infty,\ C_0' := \sup_{k\in\mathbb{N}} |\alpha_k| < \infty,$$

then for each k the forms $\mathfrak{q}_k$ and $\mathfrak{a}_{\delta,k}$ are infinitesimally $\mathfrak{a}_k$-bounded and hence the form $\mathfrak{a}_{\delta,q,k}$ is closed lower semibounded and $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k}) = \mathrm{Dom}(\mathfrak{a}_k)$ *algebraically and topologically.*

Proof. For a function $\varphi \in W_0^{1,2}([0,\infty); g_\Gamma)$, $\varphi\sqrt{g_\Gamma}$ is continuous on each interval $(t_k, t_{k+1}]$. The proof of this statement could be found in [11] and the KLMN theorem (see [17]) will be used. □

Lemma 9. *For any $k \in \mathbb{N}_0$, if the form $\mathfrak{a}_{\delta,q,k}$ is lower semibounded, the set* $\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ *is a core of the form $\mathfrak{a}_{\delta,q,k}$.*

Proof. We just prove the claim that if the form $\mathfrak{a}_{\delta,q,0}$ is lower semibounded, the set $\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,0})$ is a core of the form $\mathfrak{a}_{\delta,q,0}$. The proof of the remainder of this argument follows in a similar manner. In this proof, $D'_{\min}$ is the linear span of C^∞ functions with compact support in a single interval (t_{i-1}, t_i), $i \in \mathbb{N}$. For each function $f_i \in C_0^\infty(t_{i-1}, t_i)$, it can be extended to $[0,\infty)$. The extended function

$$\tilde{f}_i(t) = \begin{cases} f_i(t), & t \in (t_{i-1}, t_i), \\ 0, & t \in [0,\infty)\backslash(t_{i-1}, t_i) \end{cases}$$

belongs to $D'_{\min} \subset \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,0})$.

We need to show $\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,0})$ is dense in $\mathrm{Dom}(\mathfrak{a}_{\delta,q,0})$ with respect to the norm $\|\varphi\|^2_{\mathfrak{H}_{\delta,q,0}} = \mathfrak{a}_{q,0}[\varphi] + \mathfrak{a}_{\delta,0}[\varphi] + \|\varphi\|^2_{\mathfrak{H}_0}$. The method used is similar to Lemma 9 in [12]. We need to prove that for $u \in \mathrm{Dom}(\mathfrak{a}_{\delta,q,0})$ and for all $f \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,0})$,

$$(u,f)_{\mathfrak{H}_{\delta,q,0}} = \int_0^\infty u'\overline{f'}g_\Gamma + \int_0^\infty (q+1)u\overline{f}g_\Gamma + \sum_{i=1}^{\infty} \alpha_i u(t_i)\overline{f}(t_i)g_\Gamma(t_i) = 0 \tag{4.1}$$

implies that $u = 0$. The equation (4.1) holds for all $f \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,0})$, then for each interval (t_{i-1}, t_i), the equation

$$\int_{t_{i-1}}^{t_i} u'\overline{(f_i)'}g_\Gamma + \int_{t_{i-1}}^{t_i} (q+1)u\overline{(f_i)}g_\Gamma = 0$$

holds for all $f_i \in C_0^\infty(t_{i-1}, t_i)$. Then $u'' = (q+1)u$ on each interval (t_{i-1}, t_i) in the sense of distributions.

Since the equation (4.1) holds for all $f \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,0})$, integrating by parts, we get $u \in \mathrm{Dom}((\mathfrak{A}^0_{\delta,Q,0})^*)$. Then by the similar method with the Theorem 1, the only function $u \in \mathrm{Dom}(\mathfrak{a}_{\delta,q,0})$ satisfying the equation (4.1) is $u = 0$. □

Lemma 10. *For any $k \in \mathbb{N}_0$, if the form $\mathfrak{a}_{\delta,q,k}$ is lower semibounded, then it is closable. The operator associated with its closure $\overline{\mathfrak{a}_{\delta,q,k}}$ coincides with $\mathfrak{A}_{\delta,Q,k} = \mathfrak{A}^*_{\delta,Q,k}$.*

Proof. Integrating by parts, we can get that $\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k}) \subset \mathrm{Dom}(\mathfrak{a}_{\delta,q,k})$. For every function $u \in \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$,

$$\begin{aligned}\mathfrak{a}_{\delta,q,k}[u,u] &= \int_{t_k}^{\infty} |u'|^2 g_\Gamma + \int_{t_k}^{\infty} q\,|u|^2 g_\Gamma + \sum_{i=k+1}^{\infty} \alpha_i\,|u(t_i)|^2 g_\Gamma(t_i)\\ &= (\mathfrak{A}^0_{\delta,Q,k}u,u).\end{aligned}$$

If the form $\mathfrak{a}_{\delta,q,k}$ is lower semibounded, then $\mathfrak{A}^0_{\delta,Q,k}$ is lower semibounded and the form $\mathfrak{a}^0_{\delta,q,k} := \mathfrak{a}_{\delta,q,k} \restriction \mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ is closable. Since $\mathrm{Dom}(\mathfrak{A}^0_{\delta,Q,k})$ is a core of the form $\mathfrak{a}_{\delta,q,k}$, the closed form $\overline{\mathfrak{a}^0_{\delta,q,k}}$ is an extension of $\mathfrak{a}_{\delta,q,k}$, and $\overline{\mathfrak{a}^0_{\delta,q,k}} = \overline{\mathfrak{a}_{\delta,q,k}}$. The operator associated with $\overline{\mathfrak{a}^0_{\delta,q,k}}$ is Friedrichs' extension of $\mathfrak{A}^0_{\delta,Q,k}$. By Theorem 1, $\mathfrak{A}_{\delta,Q,k} = \mathfrak{A}^*_{\delta,Q,k}$, hence it is associated with $\overline{\mathfrak{a}_{\delta,q,k}}$. The proof can be proceeded for any $k \in \mathbb{N}_0$. □

5. Operators with Discrete Spectrum

5.1. **Schrödinger operators with Dirichlet boundary conditions at the root o.** In this section we extend the classical Molchanov's discreteness criterion [9] to the case of Schrödinger operator $\mathcal{L}_{\delta,Q}$ on a regular metric tree Γ. To do this we need some results given by M. Solomyak in [6]. Denote $\sigma(\mathfrak{A})$ and $\sigma_p(\mathfrak{A})$ the spectrum and the point spectrum of the operator $\mathfrak{A}$.

Lemma 11. *For the operators $\mathfrak{A}_{\delta,Q,k}$ defined in $\mathfrak{H}_k = L^2([t_k,\infty); g_\Gamma)$, $k \in \mathbb{N}_0$, we have the following results:*

(i) *If $\mathfrak{A}_{\delta,Q,0}$ is lower semibounded, then the same is true for any operator $\mathfrak{A}_{\delta,Q,k}$, $k \in \mathbb{N}$, and*

$$\min\sigma(\mathfrak{A}_{\delta,Q,0}) \leqslant \min\sigma(\mathfrak{A}_{\delta,Q,1}) \leqslant \cdots \leqslant \min\sigma(\mathfrak{A}_{\delta,Q,k}) \leqslant \cdots \tag{5.1}$$

(ii) *If the spectrum of $\mathfrak{A}_{\delta,Q,0}$ is discrete, then the same is true for any operator $\mathfrak{A}_{\delta,Q,k}$, $k \in \mathbb{N}$.*

(iii) $\sigma_p(\mathcal{L}_{\delta,Q}) = \cup_{k=0}^{\infty}\sigma_p(\mathfrak{A}_{\delta,Q,k})$*;* $\sigma(\mathcal{L}_{\delta,Q}) = \overline{\cup_{k=0}^{\infty}\sigma(\mathfrak{A}_{\delta,Q,k})}$.

Proof. (i) For each $k \in \mathbb{N}_0$, the operator $\mathfrak{A}_{\delta,Q,k}$ is the self-adjoint operator associated with the quadratic form $\mathfrak{a}_{\delta,q,k}$, then the operator $\mathfrak{A}_{\delta,Q,k}$ has the same lower bound γ_k with $\mathfrak{a}_{\delta,q,k}$ [14, pp.122-123]. Each function $\varphi \in \mathrm{Dom}(\mathfrak{a}_{\delta,q,k+1})$ can be extended to the function $\tilde\varphi$ defined on the interval $[t_k,\infty)$ by setting $\tilde\varphi(t) = 0$ for $t \in [t_k, t_{k+1})$. The extended set of $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k+1})$ is a subset of $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k})$, then we get $\gamma_{k+1} \geqslant \gamma_k$. In addition, $\mathfrak{A}_{\delta,Q,k}$ is self-adjoint, $\min\sigma(\mathfrak{A}_{\delta,Q,k}) = \gamma_k$ [15, pp.278]. Hence the statements are proved.

(ii) By Lemma 5, the spectrum $\sigma(\mathfrak{A}_{\delta,Q,k})$ is discrete if and only if the embedding $i_{\mathfrak{A}_{\delta,Q,k}}\colon \mathfrak{H}_{\mathfrak{a}_{\delta,q,k}} \hookrightarrow L^2([t_k,\infty); g_\Gamma)$ is compact, where $\mathfrak{H}_{\mathfrak{a}_{\delta,q,k}}$ denotes the space consisting of the functions in $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k})$ equipped with the norm

$$\|\varphi\|^2_{\mathfrak{a}_{\delta,q,k}} = \mathfrak{a}_{\delta,q,k}[\varphi] + (1-\gamma_k)\,\|\varphi\|^2_{L^2([t_k,\infty);g_\Gamma)}\,,\ \varphi \in \mathrm{Dom}(\mathfrak{a}_{\delta,q,k}).$$

Since the extended set of $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k+1})$ is a subset of $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k})$, the compactness of the embedding $i_{\mathfrak{A}_{\delta,Q,0}}$ implies the compactness of the embedding $i_{\mathfrak{A}_{\delta,Q,k}}$, for all $k \in \mathbb{N}$.

(iii) The statements follow from Lemma 1 (see [6]). □

Then the following is one of our main results. We prove the relationship between the spectral discreteness of the Schrödinger operator $\mathcal{L}_{\delta,Q}$ on Γ and the spectral discreteness of Schrödinger operators $\mathfrak{A}_{\delta,Q,k}$ on intervals $[t_k,\infty)$.

Theorem 2. *The spectrum of the given Schrödinger operator $\mathcal{L}_{\delta,Q}$ is discrete if and only if the following two conditions are satisfied.*
(i) *The spectrum of $\mathfrak{A}_{\delta,Q,0}$ is discrete.*
(ii) $\min\sigma(\mathfrak{A}_{\delta,Q,k})\to\infty$, *as $k\to\infty$.*

Proof. The sufficiency is obvious, so we just demonstrate the necessity. Since $\sigma(\mathcal{L}_{\delta,Q})=\overline{\cup_{k=0}^{\infty}\sigma(\mathfrak{A}_{\delta,Q,k})}$, the discreteness of $\sigma(\mathcal{L}_{\delta,Q})$ implies that the spectrum of each operator $\mathfrak{A}_{\delta,Q,k}$ is discrete, then $\sigma(\mathfrak{A}_{\delta,Q,k})=\sigma_p(\mathfrak{A}_{\delta,Q,k})$. Assume that condition (ii) is violated, then the sequence $\{\min\sigma_p(\mathfrak{A}_{\delta,Q,k})\}_{k=0}^{\infty}$ is bounded. In addition, by Lemma 11 the sequence $\{\min\sigma_p(\mathfrak{A}_{\delta,Q,k})\}_{k=0}^{\infty}$ is monotone increasing, then there must exist an accumulation point. That contradicts the discreteness of $\sigma(\mathcal{L}_{\delta,Q})$. The proof is completed. □

We turn to the spectral properties of the Schrödinger operators $\mathfrak{A}_{\delta,Q,k}$. By Lemma 7, when $q(t)\geqslant 0$ and $\{\alpha_i\}_{i=1}^{\infty}\subset[0,\infty)$, the form $\mathfrak{a}_{\delta,q,k}$ is non-negative and closed for each $k\in\mathbb{N}_0$. Then for each $k\in\mathbb{N}_0$, the operator $\mathfrak{A}_{\delta,Q,k}$ is lower semi-bounded, and $\mathfrak{A}_{\delta,Q,k}=\mathfrak{A}^*_{\delta,Q,k}$ is the associated operator with $\mathfrak{a}_{\delta,q,k}$. Following from the compact embedding theorems, we obtain a necessary and sufficient condition for $\mathfrak{A}_{\delta,Q,k}$ to have discrete spectrum in terms of quadratic forms.

Theorem 3. *Assume that $q\in L^1_{loc}([0,\infty))$, $q(t)\geqslant 0$ and $\{\alpha_i\}_{i=1}^{\infty}\subset[0,\infty)$. Then the spectrum of operator $\mathfrak{A}_{\delta,Q,0}$ is discrete if and only if for every $\epsilon>0$*

$$\int_t^{t+\epsilon}q(t)\mathrm{d}t+\sum_{t_i\in(t,t+\epsilon]}\alpha_i\to\infty\quad \textit{as } t\to\infty. \tag{5.2}$$

Proof. Sufficiency: By Lemma 7 the form $\mathfrak{a}_{\delta,q,0}$ is closed in $\mathfrak{H}=L^2([0,\infty);g_\Gamma)$. Let $\mathfrak{H}_{\delta,q,0}$ be the Hilbert space generated by $\mathfrak{a}_{\delta,q,0}$. We denote the unit ball in $\mathfrak{H}_{\delta,q,0}$ as $U_{\delta,q,0}$. Let us show that the unit ball $U_{\delta,q,0}$,

$$\{\varphi\in W_0^{1,2}([0,\infty);g_\Gamma):\quad \|\varphi\|^2_{W^{1,2}([0,\infty);g_\Gamma)}+\left\|q^{1/2}\varphi\right\|^2_{L^2([0,\infty);g_\Gamma)}+\sum_{i=1}^{\infty}\alpha_i\left|\varphi(t_i)\right|^2g_\Gamma(t_i)\leqslant 1\},$$

is compact in $L^2([0,\infty);g_\Gamma)$. Since the edge lengths of Γ have lower bound $S>0$ and upper bound M, in addition $b(v)<\infty$ for any v, the embedding

$$W^{1,2}([0,a];g_\Gamma)\hookrightarrow L^2([0,a];g_\Gamma)$$

is compact for any $a>0$. It suffices to show that $\int_N^\infty|\varphi(t)|^2g_\Gamma(t)\mathrm{d}t$ uniformly tend to zero in $U_{\delta,q,0}$.

Let us divide the interval $[0,\infty)$ into semiclosed intervals Ω'_n of lengths 2ϵ, $\Omega'_i\cap\Omega'_j=\varnothing$. $T_1:=\{\Omega'_n\}$ is a division of $[0,\infty)$. Since $\{t_i\}_{k=1}^{\infty}$ is a strictly increasing sequence, such that $t_i\to\infty$ as $i\to\infty$ and $|t_{i+1}-t_i|\geqslant S>0$, let

$$T_2:=\{I_1:=[t_0,t_1]\}\cup\{I_i:=(t_{i-1},t_i],i=2,3\ldots\}$$

be another division of $[0,\infty)$. Let $T := T_1 + T_2$ (this means we unite the dividing points of T_1 and T_2) and denote T as $\{\Omega_n\}$. Then g_Γ is a constant function in a given Ω_n. For any $\varphi \in W^{1,2}([0,\infty); g_\Gamma)$ and any x, $y \in \Omega_n$, we have

$$\begin{aligned} \left|\varphi^2(x)g_\Gamma(x) - \varphi^2(y)g_\Gamma(y)\right| &= \left|\varphi^2(x) - \varphi^2(y)\right| g_\Gamma(x) \\ &= 2\left|\int_x^y \varphi(t)\varphi'(t)\mathrm{d}t\right| g_\Gamma(x) \\ &\leqslant 2\int_{\Omega_n} |\varphi(t)|\,|\varphi'(t)|\, g_\Gamma(t)\mathrm{d}t \\ &\leqslant \|\varphi\|^2_{W^{1,2}(\Omega_n;g_\Gamma)}. \end{aligned}$$

Since $\varphi\sqrt{g_\Gamma}$ is continuous on Ω_n, there exists $y_n \in \Omega_n$, such that

$$\int_{\Omega_n} q\,|\varphi|^2\, g_\Gamma + \sum_{t_i\in\Omega_n} \alpha_i\,|\varphi(t_i)|^2\, g_\Gamma(t_i) = |\varphi(y_n)|^2\, g_\Gamma(y_n)(\int_{\Omega_n} q + \sum_{t_i\in\Omega_n} \alpha_i).$$

Then we obtain

$$\begin{aligned} \int_{\Omega_n} |\varphi(t)|^2\, g_\Gamma(t)\mathrm{d}x &\leqslant 2\epsilon\left|\varphi^2(y_n)g_\Gamma(y_n)\right| + 2\epsilon\,\|\varphi\|^2_{W^{1,2}(\Omega_n;g_\Gamma)} \\ &\leqslant 2\epsilon(\int_{\Omega_n} q(t)\,|\varphi(t)|^2\, g_\Gamma(t)\mathrm{d}t + \sum_{t_i\in\Omega_n,} \alpha_i\,|\varphi(t_i)|^2\, g_\Gamma(t_i)) \\ &\quad\cdot(\int_{\Omega_n} q(t)\mathrm{d}t + \sum_{t_i\in\Omega_n} \alpha_i)^{-1} + 2\epsilon\,\|\varphi\|^2_{W^{1,2}(\Omega_n;g_\Gamma)}. \end{aligned} \tag{5.3}$$

According to condition (5.2), there exists $N \in \mathbb{N}$, such that

$$\int_{\Omega_n} q(t)\mathrm{d}t + \sum_{t_i\in\Omega_n} \alpha_i > 1 \text{ for all } n \geqslant N. \tag{5.4}$$

Combining (5.3) with (5.4), we get

$$\begin{aligned} \int_{y_n}^{\infty} |\varphi(t)|^2\, g_\Gamma(t)\mathrm{d}x &\leqslant 2\epsilon\sum_{n=1}^{\infty}(\int_{\Omega_n} q(t)\,|\varphi(t)|^2\, g_\Gamma(t)\mathrm{d}t + \sum_{t_i\in\Omega_n} \alpha_i\,|\varphi(t_i)|^2\, g_\Gamma(t_i)) \\ &\quad +2\epsilon\,\|\varphi\|^2_{W^{1,2}([0,\infty);g_\Gamma)}, \end{aligned}$$

i.e.,

$$\int_{y_n}^{\infty} |\varphi(t)|^2\, g_\Gamma(t)\mathrm{d}x \leqslant 2\epsilon.$$

Hence by Lemma 5, the spectrum of $\mathfrak{A}_{\delta,Q,0}$ is discrete.

Necessity: We need a new division of $[0,\infty)$ to ensure the lengths of intervals in the division have a uniform positive lower bound. We have a natural partition

$$T_2 := \{I_1 := [t_0, t_1]\} \cup \{I_i := (t_{i-1}, t_i],\ i = 2, 3\ldots\},$$

and $S \leqslant |t_i - t_{i-1}| \leqslant M$. For each interval I_i we divide it into N equal parts. We unite the dividing points of all I_i and T_2, then we get the division

$$T_N := \{(x_n, x_{n+1}]\}_{n=2}^{\infty} \cup \{[x_1, x_2]\},\ x_1 = t_0 = 0,$$

which relies on the number N. Assume that condition (5.2) is violated. Then there exists N_0 and a sequence $\{x_{n_j}\}$ satisfies $x_{n_j} \to \infty$, such that the following inequality

$$\int_{x_{n_j}}^{x_{n_j}+\frac{M}{N_0}} q(t)\mathrm{d}t + \sum_{t_i\in(x_{n_j},x_{n_j}+\frac{M}{N_0}]} \alpha_k \leqslant C_1 < \infty$$

holds with some $C_1 > 0$. Let $\psi \in W^{1,2}([0,\infty);g_\Gamma)$ with $\|\psi\|_{W^{1,2}([0,\infty);g_\Gamma)} = 1$, supp $\psi \subset (0,\frac{S}{N_0})$ and $\sup_{t\in[0,\infty)}|\psi(t)| =: C_2 < +\infty$. Since $(0,\frac{S}{N_0}) \subset (t_0,t_1)$, then $g_\Gamma(t) \equiv 1$ on $(0,\frac{S}{N_0})$ and

$$\int_0^{\frac{S}{N_0}} |\psi'(t)|^2 + |\psi(t)|^2 = 1.$$

Let

$$\psi_{n_j}(t) := \psi[\frac{(t-x_{n_j})S}{(x_{n_j+1}-x_{n_j})N_0}](g_\Gamma(x_{n_j+1}))^{-1/2},$$

then

$$\begin{aligned}
\left\|\psi_{n_j}\right\|^2_{W^{1,2}([0,\infty);g_\Gamma)} &= \int_{x_{n_j}}^{x_{n_j+1}} \left|\psi_{n_j}(t)\right|^2 g_\Gamma(x_{n_j+1}) + \left|\psi'_{n_j}(t)\right|^2 g_\Gamma(x_{n_j+1})\mathrm{d}x \\
&= \int_0^{\frac{S}{N_0}} \psi^2(\xi)\frac{N_0(x_{n_j+1}-x_{n_j})}{S}\mathrm{d}\xi \\
&\quad + \int_0^{\frac{S}{N_0}} \left|\psi'(\xi)\frac{S}{N_0(x_{n_j+1}-x_{n_j})}\right|^2 \frac{N_0(x_{n_j+1}-x_{n_j})}{S}\mathrm{d}\xi \\
&\leqslant \frac{N_0(x_{n_j+1}-x_{n_j})}{S}\int_0^{\frac{S}{N_0}} \psi^2(\xi) + |\psi'(\xi)|^2\mathrm{d}\xi \\
&\leqslant \frac{M}{S},
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{a}_{\delta,q,0}[\psi_{n_j}] + \left\|\psi_{n_j}\right\|^2_{L^2([0,\infty);g_\Gamma)} \\
=\ & \int_0^\infty \left(\left|\psi'_{n_j}\right|^2 g_\Gamma + q\left|\psi_{n_j}\right|^2 g_\Gamma + \left|\psi_{n_j}\right|^2 g_\Gamma\right) + \sum_{i=1}^\infty \alpha_i \left|\psi_{n_j}(t_i)\right|^2 g_\Gamma(t_i) \\
\leqslant\ & \frac{M}{S} + \int_{x_{n_j}}^{x_{n_j+1}} q\left|\psi_{n_j}\right|^2 g_\Gamma + \sum_{t_i\in(x_{n_j},x_{n_j+1}]} \alpha_i\left|\psi_{n_j}(t_i)\right|^2 g_\Gamma(t_i) \\
\leqslant\ & \frac{M}{S} + C_2^2\int_{x_{n_j}}^{x_{n_j+1}} q + C_2^2 \sum_{t_i\in(x_{n_j},x_{n_j+1}]} \alpha_i \\
\leqslant\ & \frac{M}{S} + C_2^2 C_1.
\end{aligned}$$

Thus the sequence $\{\psi_{n_j}\}_{j=1}^{\infty}$ is bounded in $\mathfrak{H}_{\delta,q,0}$, but it is not compact in $\mathfrak{H} = L^2([0,\infty); g_\Gamma)$, since

$$\begin{aligned}
\left\|\psi_{n_j}\right\|^2_{L^2([0,\infty);g_\Gamma)} &= \int_{x_{n_j}}^{x_{n_j+1}} \left|\psi_{n_j}(t)\right|^2 g_\Gamma(x_{n_j+1})\mathrm{d}x \\
&= \int_0^{\frac{S}{N_0}} \psi^2(\xi)\frac{N_0(x_{n_j+1}-x_{n_j})}{S}\mathrm{d}\xi \\
&= \frac{N_0(x_{n_j+1}-x_{n_j})}{S}\left\|\psi\right\|^2_{L^2([0,\infty);g_\Gamma)} \\
&\geqslant \left\|\psi\right\|^2_{L^2([0,\infty);g_\Gamma)} .
\end{aligned}$$

By Lemma 5, the spectrum $\sigma(\mathfrak{A}_{\delta,Q,0})$ is not discrete. This leads to a contradiction. □

Theorem 4. *If $q \in L^1_{loc}([0,\infty))$ and $\{\alpha_k\}_{k=1}^{\infty}$ satisfy condition* (5.2) *and the operators $\mathfrak{A}_{\delta,Q,k}$ $(k \in \mathbb{N}_0)$ are self-adjoint, then* $\min\sigma(\mathfrak{A}_{\delta,Q,k}) \to \infty$, *as* $k \to \infty$.

Proof. For each $N \in \mathbb{N}$, we can define the division T_N of $[0,\infty)$ which has been introduced in the proof of Theorem 3. By the condition (5.2), for a given $N_0 \in \mathbb{N}$ and $T_{N_0} := \{[x_1,x_2]\} \cup \{(x_n,x_{n+1}]\}_{n=2}^{\infty}$, there exists t_{k_0} such that

$$\int_{x_n}^{x_n+\frac{S}{N_0}} q(t)\mathrm{d}t + \sum_{t_k\in(x_n,x_n+\frac{S}{N_0}]} \alpha_k > 1, \text{ for all } x_n \geqslant t_{k_0}.$$

For any interval $(x_n,x_{n+1}]$ satisfies $x_n \geqslant t_{k_0}$ and any function $\varphi \in \mathrm{Dom}(\mathfrak{a}_{\delta,q,k_0})$, let the 2ϵ in the inequations (5.3) equal $\frac{S}{N_0}$, we have

$$\begin{aligned}
\int_{x_n}^{x_{n+1}} \varphi^2(t)g_\Gamma(t)\mathrm{d}t \leqslant\ & \frac{M}{N_0}\int_{x_n}^{x_{n+1}} q(t)\left|\varphi(t)\right|^2 g_\Gamma(t)\mathrm{d}t \\
&+\frac{M}{N_0}\sum_{t_k\in(x_n,x_{n+1}],} \alpha_k\left|\varphi(t_k)\right|^2 g_\Gamma(t_k) \\
&+\frac{M}{N_0}\left\|\varphi\right\|^2_{L^2((x_n,x_{n+1}];g_\Gamma)} \\
&+\frac{M}{N_0}\left\|\varphi'\right\|^2_{L^2((x_n,x_{n+1}];g_\Gamma)} .
\end{aligned}$$

Then we get

$$\begin{aligned}
\int_{t_{k_0}}^{\infty} \varphi^2(t)g_\Gamma(t)\mathrm{d}t \leqslant\ & \frac{M}{N_0}\int_{t_{k_0}}^{\infty} q(t)\left|\varphi(t)\right|^2 g_\Gamma(t)\mathrm{d}t \\
&+\frac{M}{N_0}\sum_{t_k>t_{k_0}} \alpha_k\left|\varphi(t_k)\right|^2 g_\Gamma(t_k) \\
&+\frac{M}{N_0}\left\|\varphi\right\|^2_{L^2([t_{k_0},\infty);g_\Gamma)} \\
&+\frac{M}{N_0}\left\|\varphi'\right\|^2_{L^2([t_{k_0},\infty);g_\Gamma)} ,
\end{aligned}$$

which means that for a given $N_0 \in \mathbb{N}$, we could find a k_0 such that

$$(\mathfrak{A}_{\delta,Q,k_0}\varphi,\varphi)_{L^2([t_{k_0},\infty);g_\Gamma)} \geqslant \frac{N_0-M}{M}(\varphi,\varphi)_{L^2([t_{k_0},\infty);g_\Gamma)}. \tag{5.5}$$

If the operators $\mathfrak{A}_{\delta,Q,k}$ $(k \in \mathbb{N}_0)$ are self-adjoint, from [15, pp.278] it follows that $\min \sigma(\mathfrak{A}_{\delta,Q,k}) = \gamma_k$ for γ_k is the largest number γ with the property

$$(\mathfrak{A}_{\delta,Q,k}\varphi, \varphi)_{L^2([t_k,\infty);g_\Gamma)} \geqslant \gamma(\varphi, \varphi)_{L^2([t_k,\infty);g_\Gamma)},\ \varphi \in \mathrm{Dom}(\mathfrak{A}_{\delta,Q,k}).$$

Then $\min \sigma(\mathfrak{A}_{\delta,Q,k_0}) \geqslant \frac{N_0}{M} - 1$, for M is a fixed number. Together with (5.1), we get

$$\min \sigma(\mathfrak{A}_{\delta,Q,k}) \to \infty, \text{ as } k \to \infty.$$

□

Define q_- as $q_-(t) = \frac{q(t)-|q(t)|}{2}$, q_+ as $q_+(t) = \frac{q(t)+|q(t)|}{2}$. And Define α_k^- as $\alpha_k^- = \frac{\alpha_k - |\alpha_k|}{2}$, α_k^+ as $\alpha_k^+ := \frac{\alpha_k+|\alpha_k|}{2}$.

Theorem 5. *For the symmetric potential function Q, $q(t) = Q(x)$ for $t = |x|$, if $q \in L^1_{loc}([0,\infty))$ and*

$$\sup_{k\in\mathbb{N}_0} \int_{t_k}^{t_{k+1}} |q_-(t)|\,\mathrm{d}t < \infty,\ \sup_{k\in\mathbb{N}_0} \left|\alpha_k^-\right| < \infty, \tag{5.6}$$

then the operator $\mathcal{L}_{\delta,Q}$ is lower semibounded and self-adjoint. The spectrum $\sigma(\mathcal{L}_{\delta,Q})$ is discrete if and only if for every $\epsilon > 0$

$$\int_t^{t+\epsilon} q(t)\mathrm{d}t \to \infty,\ as\ t \to \infty. \tag{5.7}$$

Proof. Denote $\mathfrak{a}_{\delta_+,q_+,k}$ as the quadratic form with the function q_+ and the sequence $\{\alpha_k^+\}_{k=1}^\infty$. By Lemma 8, if q_- and sequence $\{\alpha_k^-\}_{k=1}^\infty$ satisfy the condition (5.6) the form $\mathfrak{q}_-$ and $\mathfrak{a}_{\delta_-,k}$ is $\mathfrak{a}_k$-bounded and each operator $\mathfrak{A}_{\delta,Q,k}$ with the potential q and sequence $\{\alpha_k\}_{k=1}^\infty$ is self-adjoint and lower semibounded. Moreover, $\mathrm{Dom}(\mathfrak{a}_{\delta,q,k}) = \mathrm{Dom}(\mathfrak{a}_{\delta_+,q_+,k})$ algebraically and topologically. By Theorem 3, the operator $\mathfrak{A}^+_{\delta,Q,k}$ with the potential q_+ and sequence $\{\alpha_k^+\}_{k=1}^\infty$ has discrete spectrum if and only if q_+ and $\{\alpha_k^+\}_{k=1}^\infty$ satisfy the condition (5.2). Assume q and $\{\alpha_k\}_{k=1}^\infty$ satisfy the conditions (5.6) and (5.2), then q_+ and $\{\alpha_k^+\}$ satisfy the condition (5.2) simultaneously. Along with Theorem 4 we get that if the potential q and sequence $\{\alpha_k\}_{k=1}^\infty$ satisfy the condition (5.6), $\sigma(\mathcal{L}_{\delta,Q})$ is discrete if and only if for every $\epsilon > 0$

$$\int_t^{t+\epsilon} q(t)\mathrm{d}t + \sum_{t_k\in(t,t+\epsilon]} \alpha_k \to \infty \quad \text{as } t \to \infty.$$

Next we replace the condition (5.2) with (5.7). Sufficiency is immediately from the above proof. Next we prove the necessity. Without loss of generality we can assume that $q(t) \geqslant 1$, $t \in [0,\infty)$. The length of edges in Γ has a positive lower bound S, we let $\varepsilon < S$. According to condition (5.2), with $\epsilon/2$ for any $C > 0$ there is $t_0 > 0$, such that

$$\int_t^{t+\epsilon/2} q(t)\mathrm{d}t + \sum_{t_k\in(t,t+\epsilon/2]} \alpha_k > C \text{ for } t > t_0.$$

Hence either $\int_t^{t+\epsilon/2} q(t)\mathrm{d}t > C$ or $\int_{t+\epsilon/2}^{t+\epsilon} q(t)\mathrm{d}t > C$ is established, since at least one of the intervals $(t, t+\epsilon/2)$ and $(t+\epsilon/2, t+\epsilon)$ contains no points of t_k. Then

$$\int_t^{t+\epsilon} q(t)\mathrm{d}t > C \text{ for } t > t_0,$$

and this completes the proof. □

5.2. **Schrödinger operators with general self-adjoint boundary conditions.** In this subsection we remove the assumption $b_0 = 1$. Consider the symmetric operator $\mathcal{L}f = L_Q f$ whose domain is

$$\begin{aligned}\mathrm{Dom}(\mathcal{L}) = \{f \in L^2_{\mathrm{comp}}(\Gamma) :\ & f \text{ is smooth on each edge, } f, f' \text{ vanish at } o,\\ & f \text{ satisfies the } \delta\text{-type conditions (2.2) at the inner vertices}\}.\end{aligned}$$

Obviously, $\mathcal{L}_{\min} \subset \mathcal{L} \subset \mathcal{L}^0_{\delta,Q}$. Infact, the operator $\mathcal{L}$ is the minimal operator whose domain consists of functions satisfying the δ-type conditions (2.2) at the inner vertices. By the same method as the reduction of $\mathcal{L}^0_{\delta,Q}$, $\mathcal{L}$ can be reduced into the direct sum of the auxiliary differential operators,

$$\mathcal{L} \sim \mathfrak{B}^{[b_0]}_{\delta,Q,0} \oplus \sum_{k=1}^{\infty} \oplus \mathfrak{B}^{[b_0 b_1 \ldots b_{k-1}(b_k-1)]}_{\delta,Q,k},$$

in which the operators $\mathfrak{B}_{\delta,Q,k}$ act in the spaces $L^2([t_k,\infty); g_\Gamma)$, respectively, with domain

$$\begin{aligned}\mathrm{Dom}(\mathfrak{B}_{\delta,Q,k}) = \{\varphi \in L^2_{\mathrm{comp}}([t_k,\infty), g_\Gamma) :\ & \varphi(t_k) = 0,\ \varphi \in C^\infty[t_{i-1}, t_i],\\ & -\varphi'' + q\varphi \in L^2([t_k,\infty), g_\Gamma),\ \varphi(t_i+) = \varphi(t_i-),\\ & b_i\varphi'(t_i+) - \varphi'(t_i-) = \alpha_i \varphi(t_i), \text{ for all } i > k\},\end{aligned}$$

for $k \in \mathbb{N}$, and

$$\begin{aligned}\mathrm{Dom}(\mathfrak{B}_{\delta,Q,0}) = \{\varphi \in L^2_{\mathrm{comp}}([0,\infty), g_\Gamma) :\ & \varphi, \varphi' \text{ vanish at } 0,\ \varphi \in C^\infty[t_{i-1}, t_i],\\ & -\varphi'' + q\varphi \in L^2([0,\infty), g_\Gamma),\ \varphi(t_i+) = \varphi(t_i-),\\ & b_i\varphi'(t_i+) - \varphi'(t_i-) = \alpha_i \varphi(t_i), \text{ for all } i \in \mathbb{N}\}.\end{aligned}$$

Recall that the deficiency indices for the symmetric operator $\mathcal{L}$ are the dimensions of the deficiency subspaces $N(\mathcal{L}^* - \lambda I)$ for λ with positive and negative imaginary part respectively. Since Q is real-valued, the symmetric operators $\mathfrak{B}_{\delta,Q,k} (k \in \mathbb{N}_0)$ have equal deficiency indices. If $\mathcal{L}$ is lower semibounded, the self-adjointness of $\mathfrak{A}_{\delta,Q,k}$ shows that the deficiency indices of $\mathfrak{B}_{\delta,Q,k}$ are $(0,0)$ if $k \in \mathbb{N}$, and are $(1,1)$ if $k = 0$. A simple calculation shows that the deficiency indices of $\mathcal{L}$ are (b_0, b_0).

Then the following lemma are valid.

Lemma 12. *All self-adjoint extensions of a symmetric operator A with finite and equal deficiency indices have the same essential spectrum.*

Proof. See [19]. □

We have the following conclusion.

Theorem 6. *Let $\tilde{\mathcal{L}}$ be an arbitrary self-adjoint extension of the operator $\mathcal{L}$. For the symmetric potential function Q, $q(t) = Q(x)$ for $t = |x|$, if $q \in L^1_{loc}[0,\infty)$ and*

$$\sup_{k \in \mathbb{N}_0} \int_{t_k}^{t_{k+1}} |q_-(t)|\,\mathrm{d}t < \infty,\ \sup_{k \in \mathbb{N}_0} \left|\alpha_k^-\right| < \infty,$$

then $\sigma(\tilde{\mathcal{L}})$ is discrete if and only if for every $\epsilon > 0$

$$\int_t^{t+\epsilon} q(t)\mathrm{d}t \to \infty,\ as\ t \to \infty.$$

Remark 1. *Let Γ be a regular tree with $h_\Gamma = \infty$. For the Laplacian Δ with Kirchhoff conditions at the inner vertices and Dirichlet condition at the root o, in* [6] *Solomyak has given the criterions for the Laplacian Δ on Γ to be positive definite and to have discrete spectrum as follows:*

(i) The Laplacian on Γ is positive definite if and only if $L_\Gamma := \int_0^{h_\Gamma} \frac{\mathrm{d}\tau}{g_\Gamma(\tau)} < \infty$ and $B(\Gamma) := \sup\limits_{t>0}(\int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}) < \infty$.

(ii) The Laplacian on Γ has discrete spectrum if and only if $L_\Gamma < \infty$, $B(\Gamma) < \infty$ and

$$\lim_{t\to\infty}(\int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}) = 0. \tag{5.8}$$

But for Γ satisfies $S \leqslant |e| \leqslant M$ for all $e \in \mathcal{E}(\Gamma)$, the condition (5.8) *can not be satisfied.*

Proof. Without loss of generality, for a regular tree Γ we assume that $b_0 = 1$, then

$$g_\Gamma(t) = \begin{cases} 1, & 0 \leqslant t \leqslant t_1, \\ b_1, & t_1 < t \leqslant t_2, \\ \cdots \\ b_1 b_2 \dots b_n, & t_n < t \leqslant t_{n+1}, \\ \cdots. \end{cases}$$

Let $\hat{t}_k := \frac{t_k + t_{k+1}}{2}$, then

$$\begin{aligned} & \int_0^{\hat{t}_k} g_\Gamma(\tau)\mathrm{d}\tau \int_{\hat{t}_k}^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)} \\ \geqslant\ & \frac{S^2}{4}(2 + \cdots + 2b_1 b_2 \cdots b_{k-2} + b_1 b_2 \cdots b_{k-1}) \\ & \cdot(\frac{1}{b_1 b_2 \cdots b_{k-1}} + \frac{2}{b_1 b_2 \cdots b_{k-1} b_k} + \cdots) \\ =\ & \frac{S^2}{4}(1 + \frac{2}{b_{k-1}} + \cdots + \frac{2}{b_1 b_2 \cdots b_{k-1}}) \cdot (1 + \frac{2}{b_k} + \frac{2}{b_k b_{k+1}} + \cdots) \\ >\ & \frac{S^2}{4}. \end{aligned}$$

That means there exists a sequence $\{\hat{t}_k\}_{k=1}^\infty \subset [0, \infty)$, $\hat{t}_k \to \infty$ as $k \to \infty$, such that $\lim\limits_{\hat{t}_k\to\infty}(\int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}) > \frac{S^2}{4}$. □

It follows that for regular tree Γ satisfying $h_\Gamma = \infty$ and $S \leqslant |e| \leqslant M$ for all $e \in \mathcal{E}(\Gamma)$, the spectrum of the Laplacian Δ could not be discrete. However for a perturbed operator $\mathcal{L}_Q = \Delta + Q$ defined on Γ the spectrum will be discrete when Q satisfies the conditions in the Theorem 5. In the following, we give two examples to illustrate this fact.

Example 1. *Consider the tree $\Gamma = \Gamma_{1,2}$ with $t_n = n$ and $b_n = 2$ for $n \in \mathbb{N}$, and $b_0 = 1$. So all the edges of $\Gamma_{1,2}$ are of length 1. We have*

$$g_{\Gamma_{1,2}}(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1, \\ 2, & 1 < t \leqslant 2, \\ \cdots \\ 2^{n-1}, & n-1 < t \leqslant n, \\ \cdots, \end{cases}$$

then $L_{\Gamma_{1,2}} = \sum_{n=1}^{\infty} (1/2)^{n-1} = 2$, and for $n-1 < t \leqslant n$,

$$\begin{aligned} \int_0^t g_{\Gamma_{1,2}}(t) \mathrm{d}t \int_t^{\infty} \frac{\mathrm{d}t}{g_{\Gamma 1,2}(t)} &< \int_0^n g_{\Gamma_{1,2}}(t) \mathrm{d}t \int_{n-1}^{\infty} \frac{\mathrm{d}t}{g_{\Gamma 1,2}(t)} \\ &< 4 \end{aligned}$$

can be estimated for all $n \in \mathbb{N}$. It follows from [6, 8] *that the spectrum of Δ is not discrete.*

Example 2. *Consider the same tree $\Gamma_{1,2}$ and the Schrödinger operator*

$$\mathcal{L}_Q f = -f'' + Qf$$

with Kirchhoff conditions at the inner vertices and Dirichlet condition at the root o. The potential Q satisfies that $Q(x) = q(|x|) = |x|$. Then the Theorem 5 implies that the spectrum of $\mathcal{L}_Q$ is discrete.

Similar methods also can be used for regular metric trees in case 3 which are described in introduction and other differential operators, such as Sturm-Liouville operators on which we are currently working.

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