Equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under CEV model

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Abstract

This paper considers an optimal investment problem for a defined contribution (DC) pension plan with default risk in a mean-variance framework. In the DC plan, contributions are supposed to be a predetermined amount of money as premiums and the pension funds are allowed to invest in a financial market which consists of a risk-free asset, a defaultable bond and a risky asset satisfied a constant elasticity of variance (CEV) model. Notice that a part of pension members could die during the accumulation phase, and their premiums should be withdrew. Thus, we consider the return of premiums clauses by an actuarial method and assume that the surviving members will share the difference between the return and the accumulation equally. Taking account of the pension fund size and the volatility of the accumulation, a mean-variance criterion as the investment objective for the DC plan can be formulated, and the original optimization problem can be decomposed into two sub-problems: a post-default case and a pre-default case. By applying a game theoretic framework, the equilibrium investment strategies and the corresponding equilibrium value functions can be obtained explicitly. Economic interpretations are given in the numerical simulation, which is presented to illustrate our results.

JEL classification: C61; G11; G22

Key words: DC pension plan; Default risk; Constant elasticity of variance (CEV) model; Mean-variance criterion; Time-consistency

1. Introduction

Nowadays, investment is an important element in the management of the pension plan, the research about the optimal investment problem for the pension plan has drawn increasing attention. Two principle types of pension fund in the literature have been concluded as the defined benefit (DB) plan and the defined contribution (DC) plan. In a DB plan, the benefits are fixed in advance by the sponsor while the contributions are initially set and subsequently adjusted to maintain the fund in balance. In contrast, a DC plan assumes that the contributions are fixed and benefits depend on the returns of the fund portfolio.

Owing to the demographic evolution and development of the capital market, especially the population aging problem and the longevity risk, the DC plan has been widely used in the global pension market in recent years and extensively studied in the literature. Based on the abroad practice of the DC plan in the reality, it has inspired literally hundreds of extensions and applications. For instance, Deelstra et al. (2004) presented a martingale method to study the related DC plan whereas some papers adopted a stochastic control framework to model the optimization problem for the DC plan, such as Cairns et al. (2006) and Giacinto et al. (2011) studied the optimal control

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strategies during the accumulation phase of the DC plan with different forms of utility functions. Han and Hung (2012) investigated the DC pension fund management problem with the inflation risk. He and Liang (2013) obtained the optimal investment strategy for the DC plan during the accumulation phase with the return of premiums. Sun et al. (2016) discussed the optimization problem for the DC plan under a jump-diffusion model.

Although many scholars have investigated optimization problems for the DC plans, we think that two aspects ought to be explored further. On one hand, most literature assumes a deterministic volatility, which goes against well-documented evidence, such as the volatility smile and the volatility clustering implied by option prices, to support the existence of stochastic volatility (SV), as far back as French et al. (1987) and Pagan and Schwert (1990) with detail studies of SV. CEV model provided by Cox and Ross (1976) pioneers the research of SV market, and still attracts much attention from academics. A couple of papers have studied the optimization problem for the DC plan under the CEV model and SV model. For example, Gao (2009) applied the Legendre transform and dual theory to study the DC plan for a pension member's whole life under the CEV model. Guan and Liang (2014) considered the optimal management of the DC plan in a stochastic interest rate and Heston SV model framework.

On the other hand, although optimal investment problems for the DC pension plan have been extensively studied, the credit or default risk is rarely considered in the modeling framework. However, it is a notorious fact that high yield corporate bonds have become increasingly prevalent for the institutional investors due to the considerably attractive rate of the return. Although the default risk has been understood as one of the significant trigger of the global credit crisis, defaultable bonds are still sought after because of the relative high profits. Bielecki and Jang (2006) studied a financial optimization problem including a defaultable asset to maximize the expected constant relative risk aversion (CRRA) utility from the terminal wealth. Bo et al. (2010) investigated a portfolio optimization problem with default risk to maximize the infinite-horizon expected discounted logarithm utility of consumption, where the default risk premium and the default intensity were assumed to rely on some stochastic factors. Capponi and Figueroa-López (2014) discussed a portfolio optimization problem in a defaultable market with finitely-many economical regimes and obtained the explicit optimal investment strategy with the objective of maximizing the expected logarithmic utility from the terminal wealth. Barucci and Cosso (2015) considered the optimal investment strategy with a defaultable asset and VaR constraint. Zhu et al. (2015) focused on the optimal reinsurance-investment problem in a defaultable market to maximize the expected exponential utility from the terminal wealth under the Heston model. Zhao et al. (2016) studied the optimal reinsurance-investment problem with default risk in a jump-diffusion model.

Since very few paper considers the optimization problem for the DC plan under the SV market with defaultable risk, we aim to derive an optimal investment strategy under the environment we mentioned above. Specifically, assume that the financial market consists of a risk-free asset, a defaultable bond and a risky asset described by the CEV model. The manager of the DC pension fund will make the investment decision under a mean-variance (MV) criterion based on the reality that they hope to maximize the size of the fund and minimize the risk of the accumulation when the pension members retire. Under the MV framework, the original optimization problem can be decomposed into two sub-problems: a post-default case and a pre-default case. Notice that the dynamic MV problem is a time-inconsistent problem, and most of literature derives the optimal strategy which is only optimal at the initial time, for example, Shen and Zeng (2015). Since time-consistency of strategies is important for a rational decision-maker, recently many scholars develop an equilibrium strategy for the dynamic mean-variance asset allocation problem, which is time-consistent[1]. Furthermore, the change of the fund size is affected by not only the result of the return

[1]As shown in Chen et al. (2014), for a dynamic optimization problem, if the strategy π_{t_1} is optimal for the decision-maker at some time t_1, and for any later time $t_2 > t_1$, she will follow the strategy π_{t_1} because it is still optimal at time t_2, i.e., $\pi_{t_1}(t) = \pi_{t_2}(t)$ for all $t > t_2$, then it is called a time-consistent strategy. Under the game theory framework, the derived subgame perfect Nash equilibrium strategy is a time-consistent strategy, see Björk and Murgoci (2010), Björk et al. (2014), and so on.

rate of the financial market but the mortality risk. During the accumulation phase, a part of pension members could die which leads to the phenomenon that their premiums should be withdrew and the rest surviving members share the difference between the return and the accumulation equally.

Based on the above setup, we aim to derive the equilibrium strategy under the MV criterion for the DC plan in a financial market consisting of a risk-free asset, a risky asset and a defaultable bond. Comparing with the existing literature, we think our paper have three innovations: (1) Optimal investment problem for the DC plan with default risk is considered. We find that the equilibrium value function under the pre-default situation is higher than the one under the post-default situation, which can be interpreted in this way: the difference between two cases stands for the loss in the pension manager's objective due to the default event. (2) Optimal investment problem for the DC plan with the return of premiums clauses under the CEV model is investigated. Our result shows that the return of premium mechanism reduces the fund size level. (3) Equilibrium investment strategy, which depends on the maximal age of the life table and the start age of the accumulation period, is derived explicitly.

The remainder of this paper is organized as follows. In Section 2, we describe the formulation of the DC pension fund optimization problem. In Section 3, by solving the extended Hamilton-Jacobi-Bellman (HJB) equations, we derive the equilibrium investment strategies and the corresponding equilibrium value functions for the post-default case and the pre-default case, respectively. In Section 4, numerical simulations are presented to illustrate our results. Section 5 concludes this paper.

2. General formulation

Let $(\Omega, \mathcal{G}, \mathrm{Q})$ be a complete probability space, which is endowed with the filtration $\mathcal{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ and $\mathcal{G}$ is enlarged filtration given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{Z}_t$. The filtration $\mathcal{F}_t$ is assumed to be generated by the Brownian motion $\{W(t)\}$, and $\mathcal{Z}_t$ is driven by a Poisson process representing the arrivals of defaults. The probability measure Q is a martingale probability measure of risk neutral measure which is assumed to be equivalent to a real-world probability measure P.

In the DC plan, contributions to the pension fund are supposed to be a predetermined amount of money as premiums during the accumulation phase. We assume that the premium per unit time is c and the accumulation period starts from the age ω_0 and lasts to the age $\omega_0 + T$ when the pension members retire, i.e., the length of the pension fund's accumulation period is T. To gain higher yields, the pension funds are allowed to invest in a financial market consisting of a risk-free asset, a defaultable bond and a risky asset. The price processes of the risk-free asset under the probability measure P follows

$$\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \quad S_0(0) = 1,$$

while the price process of the risky asset is described by the CEV model

$$\mathrm{d}S(t) = S(t)\left(r\mathrm{d}t + \sigma(S(t))^{\beta}\mathrm{d}W(t)\right), \quad S(0) = s_0,$$

where r_0 is the risk-free interest rate and r, $\sigma(S(t))^{\beta}$, β are the expected return rate, the instantaneous volatility and the elasticity parameter of the risky asset, respectively. $\{W(t)\}$ is a standard Brownian motion. To capture the features of the real market, we assume that $r > r_0$ and $\beta \geq 0$ regularly.

Unlike the price process of the risk-free asset and the risky asset which are given under the real-world probability measure P directly, the price process of the defaultable bond is firstly defined under the risk neutral measure Q and then will be transformed into the price process under the probability measure P. To investigate the price process of the defaultable bond, similar to Bielecki and Jang (2006), we provide the definition of the default process.

Definition 2.1. *Let τ be a nonnegative random variable, representing the default time of the company issuing the bond. A nondecreasing right continuous process which makes discrete jumps at the random time τ is called a default process. Denote a default process by $Z(t) := \mathbf{1}_{\tau\leq t}$, where $\mathbf{1}$ represents the indicator which has value one if there is a jump and zero otherwise.*

As in Jarrow and Turnbull (1995), Madan and Unal (1998), Duffie and Singleton (1999) and Driessen (2005), the default time τ can be modeled as the first arrival of a Poisson process. The intensity of the jump process is denoted by h, which measures the arrival rate of a default. Next, we define a martingale default process.

Definition 2.2. *The martingale default process is thus given by* $M(t) := Z(t) - \int_0^t (1 - Z(\nu))h\mathrm{d}\nu$. *The stochastic differential equation of* $M(t)$ *is* $\mathrm{d}M(t) = \mathrm{d}Z(t) - (1 - Z(\nu))h\mathrm{d}t$.

We first derive the dynamics of a defaultable bond price under measure Q. As shown in Bielecki and Jang (2006), there exists a defaultable zero-coupon bond with one-unit face value and the maturity date T_1. Suppose that if the default occurs, the investor recovers a fraction of the defaultable bond's market value just prior to default and then the post-default value of the defaultable bond is zero. The loss rate is denoted by $\zeta(\zeta \in [0, 1])$ and the default recovery rate is $1 - \zeta$. Denote $\delta = h^{\mathrm{Q}}\zeta$ as the risk neutral credit spread, and h^{Q} is the constant intensity of the default Poisson process under measure Q, then the price process of the defaultable bond under measure Q can be given by

$$B(t, T_1) = \mathbf{1}_{\tau>t}\mathrm{e}^{-(r_0+\delta)(T_1-t)} + \mathbf{1}_{\tau\le t}(1 - \zeta)\mathrm{e}^{-(r_0+\delta)(T_1-\tau)}\mathrm{e}^{r(t-\tau)}.$$

As the discussion in Bielecki and Jang (2006), $B(t, T_1)$ is a fictitious security rather than a real traded security, which allows us to account for the jump risk premium in the expected return of the defaultable bond, and the dynamic of the defaultable bond price under measure Q follows

$$\mathrm{d}B(t, T_1) = r_0B(t, T_1)\mathrm{d}t - \zeta\mathrm{e}^{-(r_0+\delta)(T_1-t)}\mathrm{d}M^{\mathrm{Q}}(t),$$

where $M^{\mathrm{Q}}(t)$ is a compensated jump process and Q martingale process.

Next, we change the price process of the defaultable bond from the risk neutral probability measure Q to the real-world probability measure P. The following Girsanov's theorem (see Kusuoka, 1999) is used to change of measures.

Theorem 2.3. *A probability* P *is equivalent to* Q *on* $\mathcal{G}$ *if and only if there exists progressively measurable process* ψ *and a predictable process* $\Delta > 0$ *such that*
(1) $\mathrm{E}^{\mathrm{P}}[L(T)] = 1$, *where*

$$\begin{aligned}
&L(t) = L_1(t)L_2(t),\\
&L_1(t) = \exp\left\{\int_0^t \psi(\nu)\mathrm{d}W^{\mathrm{Q}}(\nu) - \frac{1}{2}\int_0^t \psi^2(\nu)\mathrm{d}\nu\right\},\\
&L_2(t) = \exp\left\{\int_0^t \ln(\Delta)\mathrm{d}Z(\nu) - \int_0^{t\wedge\tau} h^{\mathrm{Q}}[\Delta - 1]\mathrm{d}\nu\right\}, \qquad \forall t \in [0, T].
\end{aligned}$$

(2) $\frac{\mathrm{d}P}{\mathrm{d}Q} = L(t)$.

Moreover, the process $W^{\mathrm{P}}(t) = W^{\mathrm{Q}}(t) - \int_0^t \psi(\nu)\mathrm{d}\nu$ *is a* $\mathcal{G}$*-Brownian under measure* P *and the process* $M^{\mathrm{P}}(t) = Z(t) - \int_0^t h^{\mathrm{Q}}\Delta(1 - Z(\nu))\mathrm{d}\nu$ *is a* $\mathcal{G}$*-martingale under measure* P.

Following Bielecki and Jang (2006), we denote by $1/\Delta$ the (constant) default risk premium. According to Duffie and Singleton (2003), the probability of default under the risk neutral probability measure Q is higher than that under the real-world probability measure P, i.e., $1/\Delta = h^{\mathrm{Q}}/h^{\mathrm{P}} \ge 1$, where h^{P} represents the density of the default process $\{Z(t)\}$ under the probability measure P. Furthermore, the process $\{M^{\mathrm{P}}(t)\}$ defined by $M^{\mathrm{P}}(t) := Z(t) - \int_0^t (1 - Z(\nu))h^{\mathrm{P}}\mathrm{d}\nu$ is a $(\mathcal{G}, \mathrm{P})$-martingale, which is assumed to be independent of $\{W(t)\}$. By applying Theorem 2.3 (Girsanov's Theorem, Kusuoka, 1999) for the default process and following the derivation of Bielecki and Jang (2006), we obtain the dynamic of the defaultable bond price under the probability measure P as follow

$$\mathrm{d}B(t, T_1) = B(t-, T_1)[r_0\mathrm{d}t + (1 - Z(t))(1 - \Delta)\delta\mathrm{d}t - (1 - Z(t-))\zeta\mathrm{d}M^{\mathrm{P}}(t)]. \tag{1}$$

The expected return of the defaultable bond in equation (1) consists of two components (see Yu (2002)). The first component is the return of the risk-free asset and the second is the difference between the risk neutral credit spread and the real-world credit spread when the default has not occurred by time t.

Denote by $\pi_1(t)$ and $\pi_2(t)$ as the money amount allocated in the risky asset and the defaultable bond by the the pension manager at time t, respectively, the rest is allocated in the risk-free asset. The investment strategy $\pi := \{(\pi_1(t), \pi_2(t))\}_{t\in[0,T]}$ will be applied by the pension manager at time t.

Considering that some pension members could die during the accumulation phase, the change of the DC pension fund size would be associated with the uncertainty of the mortality risk. Therefore, the fund manager could take the return of premiums clauses into account, which means part of the premiums should be withdrew and the surviving members share the difference between the return and the accumulation equally. Similar to He and Liang (2013) and Sun et al. (2016), to understand better our model, we first introduce the wealth process in the differential form with the time interval $[t, t+\frac{1}{n}]$

$$
\begin{aligned}
X^\pi\left(t+\tfrac{1}{n}\right) = \frac{1}{1-{}_{\frac{1}{n}}q_{\omega_0+t}} \Bigg\{ X^\pi(t)\Bigg[\left(1-\frac{\pi_1(t)}{X^\pi(t)}-\frac{\pi_2(t)}{X^\pi(t)}\right)\frac{S_0(t+\frac{1}{n})}{S_0(t)} \\
+\frac{\pi_1(t)}{X^\pi(t)}\frac{S(t+\frac{1}{n})}{S(t)}+\frac{\pi_2(t)}{X^\pi(t)}\frac{B(t+\frac{1}{n},T_1)}{B(t,T_1)}\Bigg]+\frac{c}{n}-act\,{}_{\frac{1}{n}}q_{\omega_0+t}\Bigg\}.
\end{aligned} \tag{2}
$$

Remark 2.4. In equation (2), $\left(1-\frac{\pi_1(t)}{X^\pi(t)}-\frac{\pi_2(t)}{X^\pi(t)}\right)\frac{S_0(t+\frac{1}{n})}{S_0(t)}$, $\frac{\pi_1(t)}{X^\pi(t)}\frac{S(t+\frac{1}{n})}{S(t)}$, $\frac{\pi_2(t)}{X^\pi(t)}\frac{B(t+\frac{1}{n},T_1)}{B(t,T_1)}$ represent the investments in the risk-free asset, risky asset and defaultable bond, respectively. $\frac{c}{n}$ represents the contributions during $[t, t+\frac{1}{n}]$. ${}_{\frac{1}{n}}q_{\omega_0+t}$ is an actuarial symbol standing for the probability that the person who is alive at the age of ω_0+t will be dead in the following $\frac{1}{n}$ time period, a is a parameter with the value 1 or 0. If $a=1$, the premiums are returned to the pension member when she is dead, whereas if $a=0$, the pension member obtains nothing. Therefore, $act\,{}_{\frac{1}{n}}q_{\omega_0+t}$ represents the premium which should be returned to the dead member from time t to time $t+\frac{1}{n}$. The coefficient $\frac{1}{1-{}_{\frac{1}{n}}q_{\omega_0+t}}$ means that after returning the premium, the difference between the return and the accumulation will be equally distributed by the surviving members.

To simplify equation (2), we denote (cf. He and Liang, 2013)

$$
\begin{aligned}
\triangle\delta_t^{\frac{1}{n}} = \left(1-\frac{\pi_1(t)}{X^\pi(t)}-\frac{\pi_2(t)}{X^\pi(t)}\right)\frac{S_0(t+\frac{1}{n})-S_0(t)}{S_0(t)} \\
+\frac{\pi_1(t)}{X^\pi(t)}\frac{S(t+\frac{1}{n})-S(t)}{S(t)}+\frac{\pi_2(t)}{X^\pi(t)}\frac{B(t+\frac{1}{n},T_1)-B(t,T_1)}{B(t,T_1)},
\end{aligned}
$$

and the conditional death probability ${}_tq_x = 1-{}_tp_x = 1-\mathrm{e}^{-\int_0^t \mu(x+\nu)\mathrm{d}\nu}$, where $\mu(t)$ is the force function of mortality at time t, and for $n\to\infty$,

$$
{}_{\frac{1}{n}}q_{\omega_0+t} = 1-\mathrm{e}^{-\int_0^{\frac{1}{n}}\mu(\omega_0+t+\nu)\mathrm{d}\nu} \approx \mu(\omega_0+t)\frac{1}{n} = O(\frac{1}{n})
$$

is satisfied. Similarly,

$$
\frac{{}_{\frac{1}{n}}q_{\omega_0+t}}{1-{}_{\frac{1}{n}}q_{\omega_0+t}} = \frac{1-\mathrm{e}^{-\int_0^{\frac{1}{n}}\mu(\omega_0+t+\nu)\mathrm{d}\nu}}{\mathrm{e}^{-\int_0^{\frac{1}{n}}\mu(\omega_0+t+\nu)\mathrm{d}\nu}} = \mathrm{e}^{\int_0^{\frac{1}{n}}\mu(\omega_0+t+\nu)\mathrm{d}\nu}-1 \approx \mu(\omega_0+t)\frac{1}{n} = O(\frac{1}{n}).
$$

Then equation (2) becomes

$$
\begin{aligned}
X^\pi\left(t+\tfrac{1}{n}\right) &= \left(X^\pi(t)(1+\triangle\delta_t^{\frac{1}{n}})+\frac{c}{n}-act\,{}_{\frac{1}{n}}q_{\omega_0+t}\right)\left(1+\frac{{}_{\frac{1}{n}}q_{\omega_0+t}}{1-{}_{\frac{1}{n}}q_{\omega_0+t}}\right) \\
&= X^\pi(t)(1+\triangle\delta_t^{\frac{1}{n}})+X^\pi(t)\mu(\omega_0+t)\frac{1}{n}+X^\pi(t)\triangle\delta_t^{\frac{1}{n}}\mu(\omega_0+t)\frac{1}{n}+\frac{c}{n}-act\mu(\omega_0+t)\frac{1}{n}+o(\frac{1}{n}).
\end{aligned} \tag{3}
$$

When $n \to \infty$, the fund size $X^{\pi}(t)$ satisfies

$$\begin{cases} \mathrm{d}X^{\pi}(t) = [r_0 X^{\pi}(t) + (r - r_0)\pi_1(t) + X^{\pi}(t)\mu(\omega_0 + t) + c - act\mu(\omega_0 + t)]\mathrm{d}t \\ \qquad\qquad + \pi_1(t)\sigma(S(t))^{\beta}\mathrm{d}W(t) + \pi_2(t)(1 - Z(t))\delta(1 - \Delta)\mathrm{d}t - \pi_2(t)(1 - Z(t-))\zeta \mathrm{d}M^{\mathrm{P}}(t), \\ X^{\pi}(0) = x_0. \end{cases} \tag{4}$$

According to the Abraham De Moivre model (cf. Kohler and Kohler (2000)), we characterize the force function of mortality $\mu(t)$ and the survival function $s(t)$ as follow

$$s(t) = 1 - \frac{t}{\omega}, \qquad \mu(t) = \frac{1}{\omega - t}, \qquad for \quad 0 \leq t < \omega,$$

where ω is the maximal age of the life table. Then equation (4) degenerates to

$$\begin{cases} \mathrm{d}X^{\pi}(t) = \left[r_0 X^{\pi}(t) + (r - r_0)\pi_1(t) + \dfrac{X^{\pi}(t)}{\omega - \omega_0 - t}\right]\mathrm{d}t + \dfrac{c[\omega - \omega_0 - (1 + a)t]}{\omega - \omega_0 - t}\mathrm{d}t \\ \qquad\qquad + \pi_1(t)\sigma(S(t))^{\beta}\mathrm{d}W(t) + \pi_2(t)(1 - Z(t))\delta(1 - \Delta)\mathrm{d}t - \pi_2(t)(1 - Z(t-))\zeta \mathrm{d}M^{\mathrm{P}}(t), \\ X^{\pi}(0) = x_0. \end{cases} \tag{5}$$

When default has occurred, i.e., $\tau \leq t$, we suppose that $B(t-, T_1) = 0$ and fix $\pi_2(t) = 0$ afterwards.

In the following part, we consider an optimal investment problem under the MV criterion over the investment horizon $[0, T]$. Similar to Bielecki and Jang (2006), Zhu et al. (2015) and some other papers, we assume throughout that $T < T_1$, where T_1 is the time of maturity of the defaultable bond.

Definition 2.5. *(Admissible strategy). For any fixed $t \in [0, T]$, a strategy $\pi = \{(\pi_1(\nu), \pi_2(\nu))\}_{\nu \in [t,T]}$ is said to be admissible if*
(1) π is $\mathcal{G}$-predictable;
(2) $\forall \nu \in [t, T]$, $\mathrm{E}[\int_t^T ((\pi_1(\nu))^2 (S(\nu))^{2\beta} + (\pi_2(\nu))^2)]\mathrm{d}\nu < +\infty$;
(3) $\forall (x, s, z) \in \mathbb{R} \times \mathbb{R} \times \{0, 1\}$, the equation (5) has an unique solution $X^{\pi}(\nu)_{\nu \in [t,T]}$ with $X^{\pi}(t) = x$, $S(t) = s$ and $Z(t) = z$;
(4) $\forall \nu \in [t, T]$, $\forall \varrho \in [1, +\infty)$ and $\forall (t, x, s, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \{0, 1\}$, $\mathrm{E}_{t,x,s,z}(\sup |X^{\pi}(\nu)|^{\varrho}) < +\infty$, where $\mathrm{E}_{t,x,s,z}[\cdot]$ is the conditional expectation given $X^{\pi}(t) = x$, $S(t) = s$ and $Z(t) = z$.

In addition, let $\Pi(t, x, s, z)$ denote the set of all admissible strategies and z denote the initial default state. $z = 1$ and $z = 0$ correspond to the post-default case $\tau > t$ and the pre-default case $\tau \leq t$, respectively. Some illustrations about the conditions that admissible strategies satisfy are given in Appendix A.

Taking account of both the pension fund size and the volatility of the accumulation, we formulate the optimal investment problem under the mean-variance criterion as follows

$$W(t, x, s, z; \pi^*) = \sup_{\pi \in \Pi(t,x,s,z)} \left\{ \mathrm{E}_{t,x,s,z}[X^{\pi}(T)] - \frac{\gamma}{2}\mathrm{Var}_{t,x,s,z}[X^{\pi}(T)] \right\}, \tag{6}$$

where γ is the risk averse coefficient. Problem (6) is time-inconsistent since there is a non-linear function of the expectation of terminal wealth in the variance term, and thus the Bellman optimality principle is not applicable. Most literatures assume the mean-variance problem in a precommitment formulation, which leads to the optimal strategies time-inconsistent. However, time-consistency can not be ignored for a rational decision-maker who hopes to capture an equilibrium strategy which is optimal at a time and still be optimal as time goes forward into a future time, i.e., equilibrium strategy is time-consistent, see among Björk and Murgoci (2010), Björk et al. (2014), Chen et al. (2014) and so on. Therefore, we aim to derive the equilibrium strategy for problem (6).

Definition 2.6. *Given any initial state $(t,x,s,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\}$, consider an admissible strategy $\pi^*(t)$. Define the following strategy*

$$\pi_\varepsilon(\nu,x,s,z)=\begin{cases}(\tilde{\pi}_1,\tilde{\pi}_2), & t\le\nu<t+\varepsilon,\\ \pi^*(\nu,x,s,z), & t+\varepsilon\le\nu<T,\end{cases}$$

where $\tilde{\pi}_1,\tilde{\pi}_2\in\mathbf{\Pi}:=\mathbb{R}\times\mathbb{R}$, $\varepsilon\in\mathbb{R}^+$. If

$$\liminf_{\varepsilon\downarrow 0}\frac{W(t,x,s,z;\pi^*)-W(t,x,s,z;\pi_\varepsilon)}{\varepsilon}\ge 0,$$

then π^* is called an equilibrium strategy and the equilibrium value function is $W(t,x,s,z;\pi^*)$ with

$$W(t,x,s,z;\pi^*)=\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)]-\frac{\gamma}{2}\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)] \tag{7}$$

According to Definition 2.6, the equilibrium strategy is time-consistent. For simplicity, we denote that $C^{1,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\})=\{\phi(t,x,s,z)|\phi(t,\cdot,\cdot,\cdot)$ is once continuously differentiable on $[0,T]$and $\phi(\cdot,x,s,\cdot)$ is twice continuously differentiable on $\mathbb{R}\times\mathbb{R}\}$. To provide the verification theorem, we define a variational operator: for $\forall\phi(t,x,s,z)\in C^{1,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\})$ and $\forall(t,x,s,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\}$, let

$$\mathcal{A}^\pi\phi(t,x,s,z)=\begin{cases}\phi_t(t,x,s,1)+\left[r_0x+(r-r_0)\pi_1+\dfrac{x}{\omega-\omega_0-t}+\dfrac{c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-t}\right]\\ \cdot\phi_x(t,x,s,1)+rs\phi_s(t,x,s,1)+\dfrac{1}{2}\pi_1^2\sigma^2s^{2\beta}\phi_{xx}(t,x,s,1)+\dfrac{1}{2}\sigma^2s^{2\beta+2}\phi_{ss}(t,x,s,1)\\ +\pi_1\sigma^2s^{2\beta+1}\phi_{xs}(t,x,s,1), \qquad z=1,\\ \phi_t(t,x,s,0)+\left[r_0x+(r-r_0)\pi_1+\dfrac{x}{\omega-\omega_0-t}+\dfrac{c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-t}+\pi_2\delta\right]\\ \cdot\phi_x(t,x,s,0)+rs\phi_s(t,x,s,0)+\dfrac{1}{2}\pi_1^2\sigma^2s^{2\beta}\phi_{xx}(t,x,s,0)+\dfrac{1}{2}\sigma^2s^{2\beta+2}\phi_{ss}(t,x,s,0)\\ +\pi_1\sigma^2s^{2\beta+1}\phi_{xs}(t,x,s,0)+[\phi(t,x-\zeta\pi_2,s,1)-\phi(t,x,s,0)]h^{\mathrm{P}}, \qquad z=0.\end{cases} \tag{8}$$

The following theorem provide verifications for the extended HJB equations in the post-default case ($z=1$) and the pre-default case ($z=0$), respectively.

Theorem 2.7. *(Verification theorem). For the post-default case ($z=1$) and pre-default case ($z=0$), if there exist two real-valued functions $V(t,x,s,z)$, $g(t,x,s,z)\in C^{1,2}([0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\})$ satisfying the following extended HJB system: $\forall(t,x,s,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\}$,*

$$\sup_{\pi\in\mathbf{\Pi}(t,x,s,z)}\left\{\mathcal{A}^\pi V(t,x,s,z)-\mathcal{A}^\pi\frac{\gamma}{2}(g(t,x,s,z))^2+\gamma g(t,x,s,z)\mathcal{A}^\pi g(t,x,s,z)\right\}=0, \tag{9}$$

$$V(T,x,s,z)=x, \tag{10}$$

$$\mathcal{A}^{\pi^*}g(t,x,s,z)=0,\quad g(T,x,s,z)=x, \tag{11}$$

$$\pi^*:=\arg\sup_{\pi\in\mathbf{\Pi}(t,x,s,z)}\left\{\mathcal{A}^\pi V(t,x,s,z)-\mathcal{A}^\pi\frac{\gamma}{2}(g(t,x,s,z))^2+\gamma g(t,x,s,z)\mathcal{A}^\pi g(t,x,s,z)\right\}, \tag{12}$$

then $W(t,x,s,z;\pi^*)=V(t,x,s,z)$, $\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)]=g(t,x,s,z)$ and π^* is the equilibrium investment strategy.

Proof. See Appendix B. □

3. Solution to the model

In this section, we derive the explicit solutions of the equilibrium investment strategy and the corresponding equilibrium value function for the DC pension plan in the post-default case ($z = 1$) and the pre-default case ($z = 0$), respectively.

In the post-default case, note that $B(t, T_1) = 0$, $\tau \le t \le T$, and thus $\pi_2(t) = 0$, $\tau \le t \le T$. Suppose that there exist two functions $V(t, x, s, 1)$ and $g(t, x, s, 1)$ satisfying the conditions given in Theorem 2.7. According to the expression of $\mathcal{A}^\pi$ in equation (8), we rewrite equation (9) as

$$\begin{aligned}\sup_{\pi\in\mathbf{\Pi}(t,x,s,1)} \Bigg\{ & V_t(t,x,s,1) + \left[r_0 x + (r - r_0)\pi_1 + \frac{x}{\omega-\omega_0-t} + \frac{c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-t} \right] V_x(t,x,s,1) \\ & + rsV_s(t,x,s,1) + \frac{1}{2}\pi_1^2\sigma^2 s^{2\beta}(V_{xx}(t,x,s,1) - \gamma(g_x(t,x,s,1))^2) \\ & + \frac{1}{2}\sigma^2 s^{2\beta+2}(V_{ss}(t,x,s,1) - \gamma(g_s(t,x,s,1))^2) \\ & + \pi_1\sigma^2 s^{2\beta+1}(V_{xs}(t,x,s,1) - \gamma g_x(t,x,s,1) g_s(t,x,s,1)) \Bigg\} = 0. \end{aligned} \tag{13}$$

For the pre-default case, suppose that there exist two functions $V(t, x, s, 0)$ and $g(t, x, s, 0)$ satisfying the conditions given in Theorem 2.7. Based on equation (8), equation (9) becomes

$$\begin{aligned}\sup_{\pi\in\mathbf{\Pi}(t,x,s,0)} \Bigg\{ & V_t(t,x,s,0) + \left[r_0 x + (r - r_0)\pi_1 + \frac{x}{\omega-\omega_0-t} + \frac{c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-t} + \pi_2\delta \right] V_x(t,x,s,0) \\ & + rsV_s(t,x,s,0) + \frac{1}{2}\pi_1^2\sigma^2 s^{2\beta}(V_{xx}(t,x,s,0) - \gamma(g_x(t,x,s,0))^2) \\ & + \frac{1}{2}\sigma^2 s^{2\beta+2}(V_{ss}(t,x,s,0) - \gamma(g_s(t,x,s,0))^2) \\ & + \pi_1\sigma^2 s^{2\beta+1}(V_{xs}(t,x,s,0) - \gamma g_x(t,x,s,0) g_s(t,x,s,0)) \\ & + [V(t,x-\zeta\pi_2,s,1) - V(t,x,s,0)]h^{\mathrm{P}} - \frac{1}{2}[g(t,x-\zeta\pi_2,s,1) - g(t,x,s,0)]^2 h^{\mathrm{P}} \Bigg\} = 0. \end{aligned} \tag{14}$$

More details are given in Appendix C, and the equilibrium strategy and the corresponding equilibrium value function are summarized in Theorem 3.1.

Theorem 3.1. *For the mean-variance problem (6), the equilibrium investment strategy is given by*

$$\pi_1^*(t) = \frac{(r-r_0)(\omega-\omega_0-T)\mathrm{e}^{-r_0(T-t)}}{\gamma\sigma^2(S(t))^{2\beta}(\omega-\omega_0-t)} \left[1 + \frac{r-r_0}{r_0}\left(1 - \mathrm{e}^{2r_0\beta(t-T)}\right)\right], \tag{15}$$

$$\pi_2^*(t) = \left[\frac{(\omega-\omega_0-T)\mathrm{e}^{r_0(T-t)}}{\gamma(\omega-\omega_0-t)} \left(\frac{\delta}{h^{\mathrm{P}}\zeta^2} + \frac{b_1(t)-b_2(t)-1}{\zeta} \right) \right] \mathbf{1}_{\tau>t}, \tag{16}$$

and the equilibrium value function is

$$V(t,x,s,z) = \begin{cases} \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x + \dfrac{A_1(t)}{\gamma}(S(t))^{-2\beta} + \dfrac{B_1(t)}{\gamma}, & z=1, \\ \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x + \dfrac{A_2(t)}{\gamma}(S(t))^{-2\beta} + \dfrac{B_2(t)}{\gamma}, & z=0. \end{cases} \tag{17}$$

Furthermore, the expectation and variance of the terminal value associated with the equilibrium investment strategy are

$$\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)] = \begin{cases} \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x + \dfrac{a_1(t)}{\gamma}(S(t))^{-2\beta} + \dfrac{b_1(t)}{\gamma}, & z=1, \\ \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x + \dfrac{a_2(t)}{\gamma}(S(t))^{-2\beta} + \dfrac{b_2(t)}{\gamma}, & z=0, \end{cases} \tag{18}$$

$$
\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)] = \begin{cases} \dfrac{2}{\gamma^2}[(a_1(t) - A_1(t))(S(t))^{-2\beta} + b_1(t) - B_1(t)], & z = 1, \\ \dfrac{2}{\gamma^2}[(a_2(t) - A_2(t))(S(t))^{-2\beta} + b_2(t) - B_2(t)], & z = 0, \end{cases} \tag{19}
$$

where $A_1(t)$, $A_2(t)$, $B_1(t)$, $B_2(t)$, $a_1(t)$, $a_2(t)$, $b_1(t)$ and $b_2(t)$ are given by

$$
\begin{aligned}
a_1(t) &= a_2(t) = \frac{(r - r_0)^2}{2r_0\beta\sigma^2}(1 - \mathrm{e}^{2r_0\beta(t-T)}), \\
b_1(t) &= \frac{(2\beta+1)(r-r_0)^2}{2r_0}(T-t) - \frac{(2\beta+1)(r-r_0)^2}{4r_0^2\beta}(1 - \mathrm{e}^{2r_0\beta(t-T)}) \\
&\quad + \frac{\gamma c[r_0(\omega - \omega_0 - (1+a)t) - (1+a)]}{r_0^2(\omega - \omega_0 - T)}\mathrm{e}^{r_0(T-t)} - \frac{\gamma c[r_0(\omega - \omega_0 - (1+a)T) - (1+a)]}{r_0^2(\omega - \omega_0 - T)}, \\
b_2(t) &= \frac{(2\beta+1)(r-r_0)^2}{2r_0}(T-t) - \frac{(2\beta+1)(r-r_0)^2}{4r_0^2\beta}(1 - \mathrm{e}^{2r_0\beta(t-T)}) \\
&\quad + \frac{\delta\gamma c[(r_0 + \frac{\delta}{\zeta})(r_0(\omega - \omega_0 - (1+a)t) - (1+a)) - r_0(1+a)]}{\zeta(r_0 + \frac{\delta}{\zeta})^2 r_0^2(\omega - \omega_0 - T)}\mathrm{e}^{r_0(T-t)} \\
&\quad - \frac{\delta\gamma c[(r_0 + \frac{\delta}{\zeta})(r_0(\omega - \omega_0 - (1+a)T) - (1+a)) - r_0(1+a)]}{\zeta(r_0 + \frac{\delta}{\zeta})^2 r_0^2(\omega - \omega_0 - T)}\mathrm{e}^{\frac{\delta}{\zeta}(t-T)} \\
&\quad - \frac{\gamma c[r_0(\omega - \omega_0 - (1+a)T) - (1+a)]}{r_0^2(\omega - \omega_0 - T)}(1 - \mathrm{e}^{\frac{\delta}{\zeta}(t-T)}) \\
&\quad + \frac{\gamma c[(r_0 + \frac{\delta}{\zeta})(\omega - \omega_0 - (1+a)t) - (1+a)]}{(r_0 + \frac{\delta}{\zeta})^2(\omega - \omega_0 - T)}\mathrm{e}^{r_0(T-t)} \\
&\quad - \frac{\gamma c[(r_0 + \frac{\delta}{\zeta})(\omega - \omega_0 - (1+a)T) - (1+a)]}{(r_0 + \frac{\delta}{\zeta})^2(\omega - \omega_0 - T)}\mathrm{e}^{\frac{\delta}{\zeta}(t-T)} + \left(\frac{\delta}{h^{\mathrm{P}}\zeta} - 2 + \frac{h^{\mathrm{P}}\zeta}{\delta}\right)(1 - \mathrm{e}^{\frac{\delta}{\zeta}(t-T)}), \\
A_1(t) &= \mathrm{e}^{2r\beta t}\int_t^T \mathrm{e}^{-2r\beta\nu}\left[2\beta(r - r_0)a_1(\nu) + \frac{(r-r_0)^2}{2\sigma^2}\right]\mathrm{d}\nu, \\
B_1(t) &= \int_t^T \left[\beta(2\beta+1)\sigma^2 A_1(\nu) + \frac{\gamma c[\omega - \omega_0 - (1+a)\nu]}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-\nu)}\right]\mathrm{d}\nu, \\
A_2(t) &= \mathrm{e}^{(2r\beta + h^{\mathrm{P}})t}\int_t^T \mathrm{e}^{-(2r\beta + h^{\mathrm{P}})\nu}\left[h^{\mathrm{P}}A_1(\nu) + 2\beta(r - r_0)a_2(\nu) + \frac{(r-r_0)^2}{2\sigma^2}\right]\mathrm{d}\nu, \\
B_2(t) &= \mathrm{e}^{h^{\mathrm{P}}t}\int_t^T \mathrm{e}^{-h^{\mathrm{P}}\nu}\left[h^{\mathrm{P}}B_1(\nu) + \beta(2\beta+1)\sigma^2 A_2(\nu) + \left(\frac{\delta}{\zeta} - h^{\mathrm{P}}\right)(b_1(\nu) - b_2(\nu))\right. \\
&\qquad \left. + \frac{\gamma c[\omega - \omega_0 - (1+a)\nu]}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-\nu)} + \frac{\delta^2}{2h^{\mathrm{P}}\zeta^2} - \frac{\delta}{\zeta} + \frac{h^{\mathrm{P}}}{2}\right]\mathrm{d}\nu.
\end{aligned} \tag{20}
$$

Proof. See Appendix C. □

Remark 3.2. According to equation (7), we have

$$
\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)] = \frac{2}{\gamma}(\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)] - W(t,x,s,z;\pi^*)). \tag{21}
$$

The variance of the terminal value in Theorem 3.1 implies

$$
\frac{1}{\gamma} = \begin{cases} \sqrt{\dfrac{\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)]}{2[(a_1(t) - A_1(t))(S(t))^{-2\beta} + b_1(t) - B_1(t)]}}, & z = 1, \\ \sqrt{\dfrac{\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)]}{2[(a_2(t) - A_2(t))(S(t))^{-2\beta} + b_2(t) - B_2(t)]}}, & z = 0. \end{cases} \tag{22}
$$

Together with equation (18), we obtain

$$\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)] = \begin{cases} \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x + \sqrt{\dfrac{\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)](a_1(t)(S(t))^{-2\beta}+b_1(t))^2}{2[(a_1(t)-A_1(t))(S(t))^{-2\beta}+b_1(t)-B_1(t)]}}, & z=1, \\ \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x + \sqrt{\dfrac{\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)](a_2(t)(S(t))^{-2\beta}+b_2(t))^2}{2[(a_2(t)-A_2(t))(S(t))^{-2\beta}+b_2(t)-B_2(t)]}}, & z=0. \end{cases} \tag{23}$$

Equation (23) is known as the efficient frontier of the investment problem at the initial state (t,x,s,z) in modern portfolio theory. The efficient frontier is also a straight line in the mean-standard deviation plane, no matter at which state.

Remark 3.3. Let

$$M(t) = \frac{(r-r_0)(\omega-\omega_0-T)}{\gamma\sigma^2(S(t))^{2\beta}(\omega-\omega_0-t)}\mathrm{e}^{-r_0(T-t)},$$

$$N(t) = 1+\frac{r-r_0}{r_0}\left(1-\mathrm{e}^{2r_0\beta(t-T)}\right),$$

then we find the equilibrium strategy invested in the risky asset $\pi_1^*(t) = M(t)\cdot N(t)$. Thus the money amount invested in the risky asset for the DC plan under the CEV model can be decomposed into two components. The first component $M(t)$, called as the moving GBM strategy, is actually the optimal investment strategy under the GBM model. The other component $N(t)$ is the hedge term for the volatility risk, which can be understood as a correction factor.

The following corollary discusses the properties of the correction factor.

Corollary 3.4. *The correction factor $N(t)$ is a monotone decreasing function with respect to time t and satisfies*

$$1 \le N(t) \le 1+\frac{r-r_0}{r_0}\left[1-\mathrm{e}^{-2r_0\beta T}\right], \qquad 0\le t\le T. \tag{24}$$

Proof. Since $r > r_0$ and $\beta > 0$, we derive

$$N_t(t) = -2\beta(r-r_0)\mathrm{e}^{2r_0\beta(t-T)} < 0,$$

which implies that the correction factor is a monotone decreasing function with respect to time t. Since

$$N(0) = 1+\frac{r-r_0}{r_0}\left[1-\mathrm{e}^{-2r_0\beta T}\right]$$

and $N(T)=1$, we obtain inequality (24). □

Corollary 3.4 shows the same insight as intuition that the pension manager will invest more money in the risky asset at the beginning of the investment horizon and steadily decrease the amount thereafter. The result can be interpreted from this perspective that during the initial stage of the accumulation phase, the death probability is so small that the new premiums received could cover the expenses caused by the return of premium clauses. As time passes, the fund size increases, which requires less investment in the risky asset to avoid risk.

Remark 3.5. The equilibrium investment strategy $\pi^* = \{(\pi_1^*(t), \pi_2^*(t))\}_{t\in[0,T]}$ depends on the risk aversion coefficient γ. Based on the expressions in equations (18) and (19), a higher γ leads to the lower expectation and variance of the terminal wealth, which means that a pension manager with higher risk-aversion level will invest less in the defaultable bond and the risky asset to avoid risk.

In the following two propositions, we present two special cases of our model.

Proposition 3.6. *(No defaultable bond case) Under the case without the defaultable security in the financial market, the equilibrium strategy becomes*

$$\pi^*(t) = \frac{(r-r_0)(\omega-\omega_0-T)\mathrm{e}^{-r_0(T-t)}}{\gamma\sigma^2(S(t))^{2\beta}(\omega-\omega_0-t)}\left[1+\frac{r-r_0}{r_0}\left(1-\mathrm{e}^{2r_0\beta(t-T)}\right)\right],$$

and the equilibrium value function is that of the post-default case, i.e.,

$$V(t,x,s)=\frac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\frac{A_1(t)}{\gamma}(S(t))^{-2\beta}+\frac{B_1(t)}{\gamma},$$

where $A_1(t)$ and $B_1(t)$ are give in equation (20).

From Proposition 3.6, we find that the equilibrium investment strategy in the case of no defaultable bond is the same as that in the post-default case as mentioned above.

Proposition 3.7. *(GBM case) If $\beta=0$, the CEV model reduces to the GBM model, the equilibrium strategy degenerates to*

$$\bar{\pi}_1^*(t)=\frac{(r-r_0)(\omega-\omega_0-T)\mathrm{e}^{-r_0(T-t)}}{\gamma\sigma^2(\omega-\omega_0-t)},$$

$$\bar{\pi}_2^*(t)=\left[\frac{(\omega-\omega_0-T)\mathrm{e}^{r_0(T-t)}}{\gamma(\omega-\omega_0-t)}\left(\frac{\delta}{h^{\mathrm{P}}\zeta^2}+\frac{\bar{b}_1(t)-\bar{b}_2(t)-1}{\zeta}\right)\right]\mathbf{1}_{\tau>t},$$

and the equilibrium value function is

$$\bar{V}(t,x,z)=\begin{cases}\dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\dfrac{\bar{B}_1(t)}{\gamma}, & z=1,\\[2ex] \dfrac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\dfrac{\bar{B}_2(t)}{\gamma}, & z=0,\end{cases}$$

where

$$\begin{aligned}
\bar{b}_1(t)=&\frac{(r-r_0)^2}{\sigma^2}(T-t)+\frac{\gamma c[r_0(\omega-\omega_0-(1+a)t)-(1+a)]}{r_0^2(\omega-\omega_0-T)}\mathrm{e}^{r_0(T-t)}\\
&-\frac{\gamma c[r_0(\omega-\omega_0-(1+a)T)-(1+a)]}{r_0^2(\omega-\omega_0-T)},\\
\bar{b}_2(t)=&\frac{(r-r_0)^2}{\sigma^2}(T-t)+\frac{\delta\gamma c[(r_0+\frac{\delta}{\zeta})(r_0(\omega-\omega_0-(1+a)t)-(1+a))-r_0(1+a)]}{\zeta(r_0+\frac{\delta}{\zeta})^2r_0^2(\omega-\omega_0-T)}\mathrm{e}^{r_0(T-t)}\\
&-\frac{\delta\gamma c[(r_0+\frac{\delta}{\zeta})(r_0(\omega-\omega_0-(1+a)T)-(1+a))-r_0(1+a)]}{\zeta(r_0+\frac{\delta}{\zeta})^2r_0^2(\omega-\omega_0-T)}\mathrm{e}^{\frac{\delta}{\zeta}(t-T)}\\
&-\frac{\gamma c[r_0(\omega-\omega_0-(1+a)T)-(1+a)]}{r_0^2(\omega-\omega_0-T)}(1-\mathrm{e}^{\frac{\delta}{\zeta}(t-T)})\\
&+\frac{\gamma c[(r_0+\frac{\delta}{\zeta})(\omega-\omega_0-(1+a)t)-(1+a)]}{(r_0+\frac{\delta}{\zeta})^2(\omega-\omega_0-T)}\mathrm{e}^{r_0(T-t)}\\
&-\frac{\gamma c[(r_0+\frac{\delta}{\zeta})(\omega-\omega_0-(1+a)T)-(1+a)]}{(r_0+\frac{\delta}{\zeta})^2(\omega-\omega_0-T)}\mathrm{e}^{\frac{\delta}{\zeta}(t-T)}+\left(\frac{\delta}{h^{\mathrm{P}}\zeta}-2+\frac{h^{\mathrm{P}}\zeta}{\delta}\right)(1-\mathrm{e}^{\frac{\delta}{\zeta}(t-T)}),\\
\bar{B}_1(t)=&\frac{(r-r_0)^2}{2\sigma^2}(T-t)+\frac{\gamma c[r_0(\omega-\omega_0-(1+a)t)-(1+a)]}{r_0^2(\omega-\omega_0-T)}\mathrm{e}^{r_0(T-t)}\\
&-\frac{\gamma c[r_0(\omega-\omega_0-(1+a)T)-(1+a)]}{r_0^2(\omega-\omega_0-T)},\\
\bar{B}_2(t)=&\mathrm{e}^{h^{\mathrm{P}}t}\int_t^T\mathrm{e}^{-h^{\mathrm{P}}\nu}\left[h^{\mathrm{P}}\bar{B}_1(\nu)+\left(\frac{\delta}{\zeta}-h^{\mathrm{P}}\right)(\bar{b}_1(\nu)-\bar{b}_2(\nu))+\frac{(r-r_0)^2}{2\sigma^2}\right.\\
&\left.+\frac{\gamma c[\omega-\omega_0-(1+a)\nu]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-\nu)}+\frac{\delta^2}{2h^{\mathrm{P}}\zeta^2}-\frac{\delta}{\zeta}+\frac{h^{\mathrm{P}}}{2}\right]\mathrm{d}\nu.
\end{aligned}$$

From Proposition 3.7, we find that the equilibrium investment strategy $\bar{\pi}_1^*(t)$ is actually the equilibrium investment strategy in He and Liang (2013), i.e., our model can reduce to the case in He and Liang (2013) with $\beta=0$.

4. Numerical simulations

In this section, numerical simulations are provided to illustrate our results. Throughout numerical analysis, unless otherwise stated, the basic parameters are given by: $r_0 = 0.03$, $r = 0.12$, $\gamma = 0.5$, $\sigma = 0.2$, $\beta = 1$, $a = 1$, $\delta = 0.01$, $\zeta = 0.5$, $h^{\mathrm{P}} = 0.005$, $c = 1$, $\omega = 100$, $\omega_0 = 20$, $T = 10$, $S(0) = 5$.

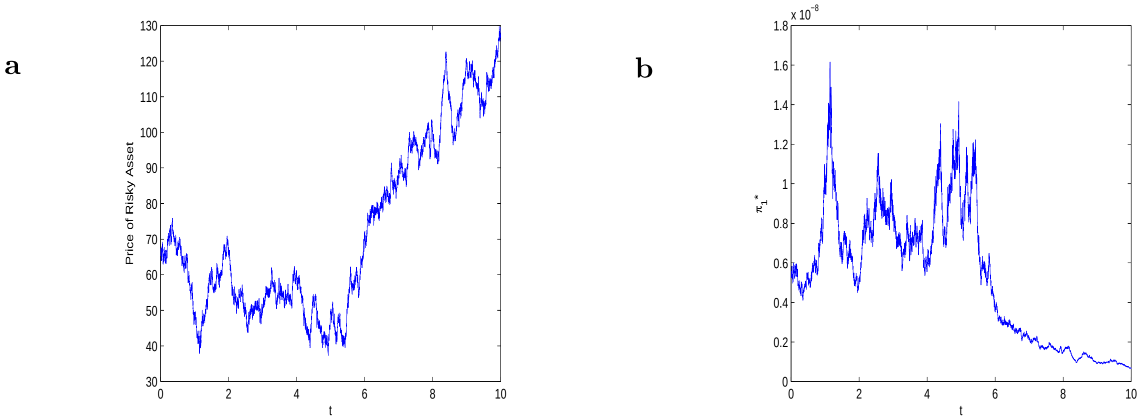

Figure 1: (a) Evolution of the risky asset's price over time. (b) Evolution of the equilibrium investment strategy invested in the risky asset.

Figure 1 plots the evolution of the risky asset's price and the dynamic behavior of the equilibrium investment strategy invested in the risky asset π_1^* over time under the CEV model. We find that the change trend of π_1^* is opposite to that of the risky asset's price. When the price process of the risky asset is more expensive, since the elasticity parameter $\beta > 0$, the volatility of the risky asset's price $\sigma(S(t))^{2\beta}$ becomes higher. Thus the pension manager will invest less wealth in the risky asset. Furthermore, notice that π_1^* decreases as time goes on, which can be explained by the reason as the death probability of the pension members is so small that the new premiums received could cover the expenses caused by the return of premium clauses at the beginning of the accumulation phase. As time passes, the fund size increases, which depresses the level of investment in the risky asset to control the risk.

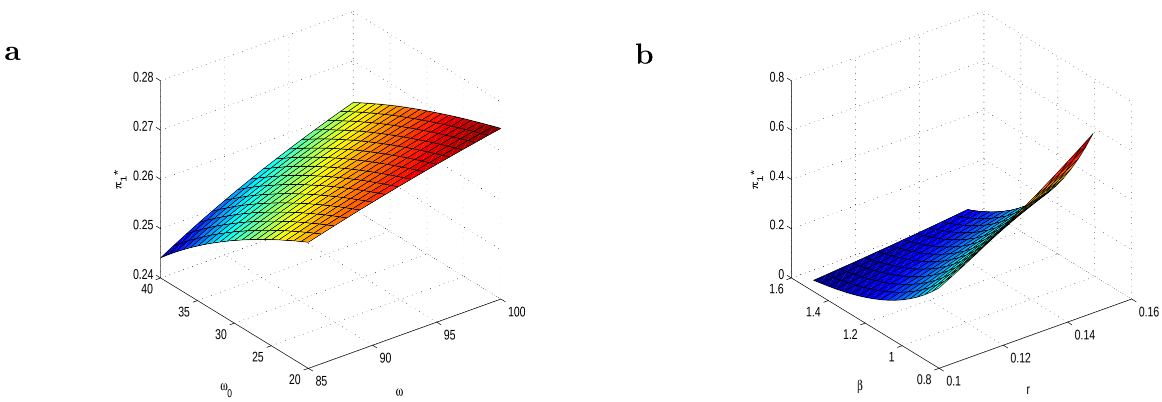

Figure 2: (a) Effects of ω and ω_0 on the equilibrium investment strategy invested in the risky asset at time 0. (b) Effects of r and β on the equilibrium investment strategy invested in the risky asset at time 0.

Figure 2 shows the impacts of ω, ω_0, r and β on the equilibrium investment strategy invested in the risky asset π_1^* at time 0. As shown in Figure 2 (a), π_1^* increases with the maximal age of the life table ω and decreases with the start age of the accumulation period ω_0. When ω_0 is fixed, a larger ω means that the pension manager predicts the death probability of the pension members in the future will be lower. Then the pension manager will invest more money in the risky asset.

Instead, if ω is fixed, a larger ω_0 implies the start age of the pension members taking part in the pension plan is older. Furthermore, the death probability of the pension members predicted by the pension manager will be higher. Therefore, the pension manager with larger ω_0 will be more cautious to the investment. In Figure 2 (b), we find that π_1^* increases with the risky asset's return r and decreases with the elasticity coefficient β. This can be explained by the fact that as r increases, the investment in the risky asset becomes more attractive. Besides, a higher β leads to a larger expected drop in volatility and an increased probability of a large adverse movement in the risky asset's price, which means the pension manager will invest less wealth in the risky asset as β increases to reduce risk from the investment in the risky asset. The results at any other time $t > 0$ are similar to that at initial time 0, so we omit explanations here.

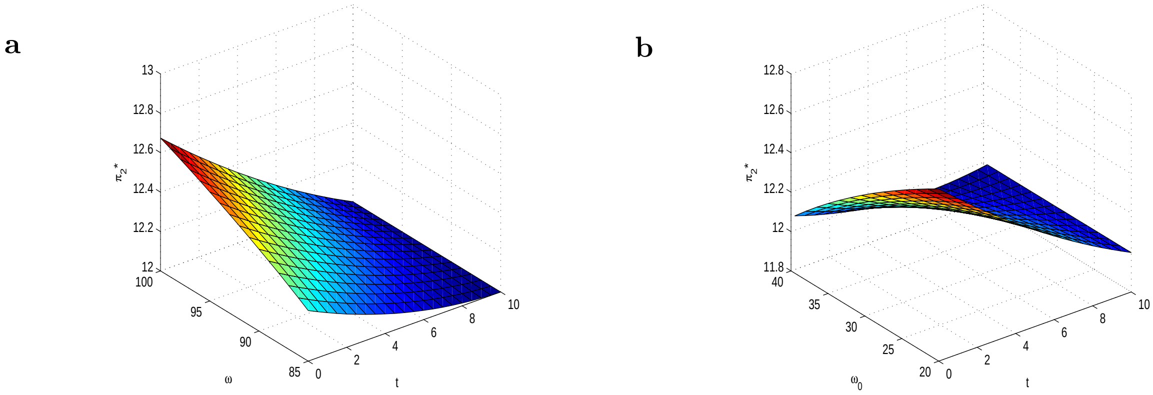

Figure 3: (a) Effect of ω on the equilibrium investment strategy invested in the defaultable bond. (b) Effect of ω_0 on the equilibrium investment strategy invested in the defaultable bond.

Figure 3 presents the impacts of ω and ω_0 on the equilibrium investment strategy invested in the defaultable bond π_2^*. We omit the analyses here due to the similarity to π_1^*.

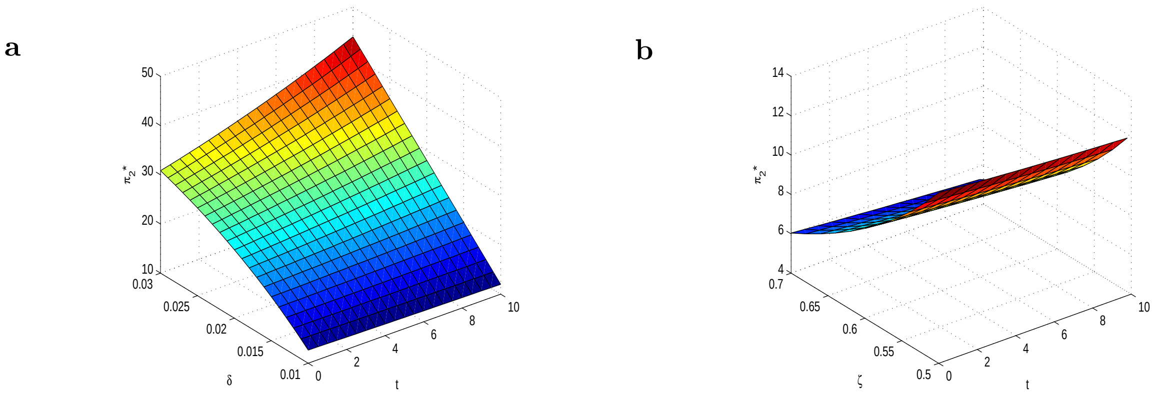

Figure 4: (a) Effect of δ on the equilibrium investment strategy in the defaultable bond. (b) Effect of ζ on the equilibrium investment strategy in the defaultable bond.

From Figure 4, we find that the equilibrium investment strategy invested in the defaultable bond π_2^* increases with the credit spread δ and decreases with the loss rate ζ. This can be attributed to the fact that the defaultable bond will be more attractive when the credit spread is higher and its default recovery rate $1 - \zeta$ is larger. Thus the pension manager will buy more defaultable bonds with higher δ and smaller ζ. Besides, a larger ζ induces a smaller recovery amount, which means that the potential loss of the pension manager becomes larger with ζ increasing. Then the pension manager will reduce the investment in the defaultable bond.

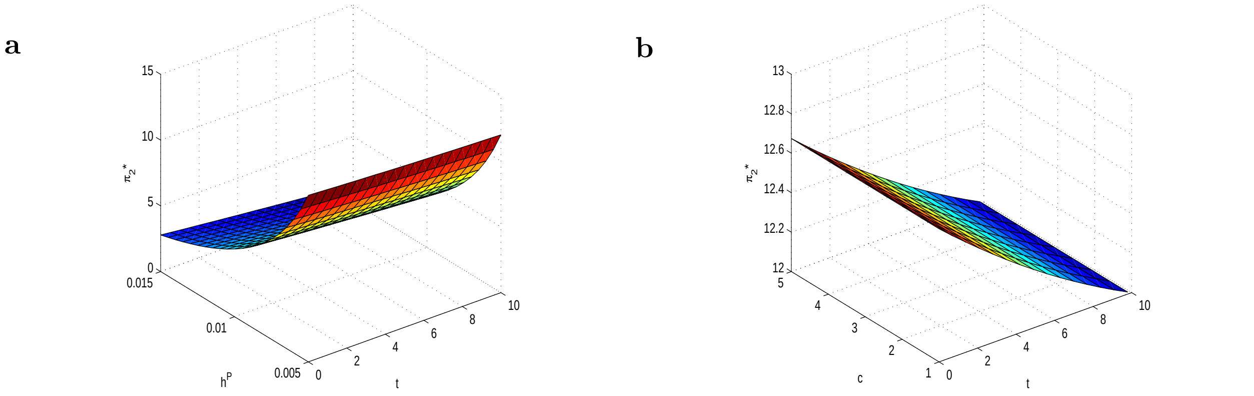

Figure 5: (a) Effect of h^{P} on the equilibrium investment strategy invested in the defaultable bond. (b) Effect of c on the equilibrium investment strategy invested in the defaultable bond.

As shown in Figure 5, the equilibrium investment strategy invested in the defaultable bond π_2^* decreases with the default intensity h^{P} and increases with premium rate paying by the pension members c. Two reasons can be explained the relationship between h^{P} and π_2^* in Figure 5: on one hand, as h^{P} increases, the probability of default becomes larger, and the counterparty risk of the defaultable bond will undermine its investment grade and thus make it less attractive to the pension manager; on the other hand, a larger h^{P} implies the less default risk premium, and the pension manager will invest less wealth in the defaultable bond. In addition, if c is higher, the pension manager will receive more premium from the pension members and she will have more wealth to invest in the financial market, which is consistent with our intuition.

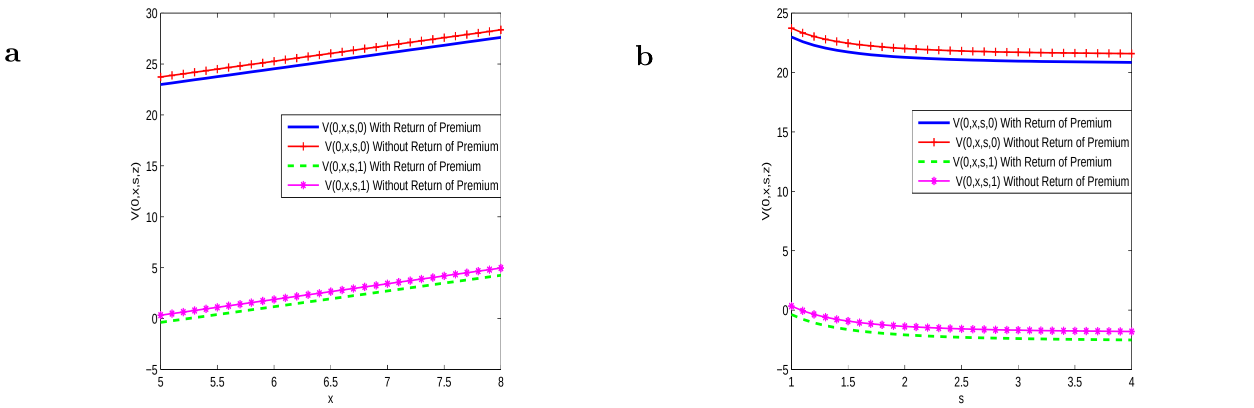

Figure 6: (a) Equilibrium value functions with respect to x at time 0. (b) Equilibrium value functions with respect to s at time 0.

Figure 6 shows the effects of x and s on the pre-default and post-default equilibrium value functions with or without premium clauses at time 0. From Figure 6, we can see that the equilibrium value functions increase with the wealth x and decreases with the price of the risky asset s. Moreover, the pre-default equilibrium value functions are higher than the post-default equilibrium value functions when all model parameters remain the same, i.e., $V(0, x, s, 0) > V(0, x, s, 1)$. The difference between $V(0, x, s, 0)$ and $V(0, x, s, 1)$ can be understood as the loss in the pension manager's objective due to the default event. We also find that the equilibrium value functions without premium clauses are higher than those with premium clauses, which can be attributed to the fact that the pension plan with premium clauses gives out part of the accumulations to the pension members who die during the accumulation phase, and this return of premium mechanism definitely reduces the fund size level. The results at any other time $t > 0$ are similar to that at initial time 0, so we omit explanations here.

5. Conclusion

This paper studies a mean-variance optimization management problem for the DC plan with default risk under the CEV model during the accumulation phase. Before retirement, contributions to the pension fund are supposed to be a predetermined amount of money as premiums. To gain higher yield, the pension funds are allowed to invest in a risk-free asset, a defaultable bond and a risky asset whose price process satisfies the CEV model. At the same time, the pension members could die during the accumulation phase which means the return of premiums clauses should also be considered. To model the above situation, assume that the premiums of the dead members can be withdrew, and the surviving members share the difference between the return and the accumulation equally. In addition, a wise fund manager considers an equilibrium strategy, which is time-consistent, and the corresponding problem could be formulated from a game theoretic perspective. First, we obtain the extended HJB equations for the post-default and the pre-default cases, respectively. By applying ansatz and variable separation, we derive the explicit expressions of the equilibrium investment strategy and the corresponding equilibrium value function for the two cases. Secondly, we analyze the properties of the equilibrium investment strategy. Finally, numerical simulations are proposed to illustrate the impacts of model parameters on the equilibrium strategy. The main findings are as follow: (1) both the maximal age of the life table ω and the start age of the accumulation period ω_0 have impacts on the money amounts invested in the defaultable bond and the risky asset, which can be attributed to that ω and ω_0 has close connections with the mortality risk; (2) the pre-default equilibrium value function is higher than the post-default equilibrium value function, which implies that the difference between two cases stands for the loss in the pension manager's objective due to the default event; (3) the equilibrium value function without premium clauses is higher than that with premium clauses, which can be interpreted by that the return of premium mechanism reduces the fund size level; (4) the efficient frontier is a straight line in the mean-standard derivation plane, which is independent of the state.

Acknowledgment

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Appendix A.
Illustrations of Definition 2.5.

To illustrate the existence and uniqueness of equation (5), we firstly show the CEV model

$$\mathrm{d}S(t) = S(t)\left(r\mathrm{d}t + \sigma(S(t))^{\beta}\mathrm{d}W(t)\right)$$

has a unique solution. Denote $m(t) = (S(t))^{-2\beta}$. By Itô's lemma, we can see

$$\mathrm{d}m(t) = (\beta(2\beta+1)\sigma^2 - 2\beta r m(t))\mathrm{d}t - 2\beta\sigma\sqrt{m(t)}\mathrm{d}W(t).$$

This is a mean-reverting square-root model. From Theorem 1.5.5.1 in Jeanblanc et al. (2009), we know that the above equation has a unique strong solution. When $\beta > 0$, the Feller condition is satisfied, i.e., $2\beta(2\beta+1)\sigma^2 > 4\beta^2\sigma^2$, so $m(t) > 0$, a.s.. When $\beta = 0$, the CEV model reduces to the geometric Brownian motion model. Therefore, for $\beta \geq 0$, CEV model has a unique solution such that $S(t) \in (0, +\infty)$, a.s.. In Dufresne (2001), it is found that all the moments of an integrated square-root process are finite. Using this result, we can further deduce that for any $\varrho \in [1, +\infty)$,

$$\begin{aligned}
\mathrm{E}\left[\sup_{\nu\in[0,T]}|m(\nu)|^{\varrho}\right] &\leq K_1\left\{1 + \mathrm{E}\left[\left(\int_0^T |m(\nu)|\mathrm{d}\nu\right)^{\varrho}\right] + \mathrm{E}\left[\sup_{t\in[0,T]}\left(\int_0^t \sqrt{m(\nu)}\mathrm{d}W(\nu)\right)^{\varrho}\right]\right\} \\
&\leq K_2\left\{1 + \mathrm{E}\left[\left(\int_0^T |m(\nu)|\mathrm{d}\nu\right)^{\varrho}\right] + \mathrm{E}\left[\left(\int_0^T |m(\nu)|\mathrm{d}\nu\right)^{\varrho/2}\right]\right\} < \infty,
\end{aligned}$$

where K_1 and K_2 are positive constants. In the above inequality, the second line follows from the Burkholder-Davis-Gundy inequality. Next, from conditions (1) and (2) in Definition 2.5 and the boundedness of parameters, we know that

$$\mathrm{E}\left[\int_t^T \left|(r-r_0)\pi(\nu)+\frac{c[\omega-\omega_0-(1+a)\nu]}{\omega-\omega_0-\nu}\right|^2 \mathrm{d}\nu\right]<\infty,$$
$$\mathrm{E}\left[\int_t^T \left(|\pi_1(\nu)\sigma(S(\nu))^\beta|^2+|\pi_2(\nu)(1-Z(\nu))\delta(1-\Delta)|^2+|\pi_2(\nu)(1-Z(\nu))\zeta|^2\right)\mathrm{d}\nu\right]<\infty.$$

Thus by Theorem 6.3 in Yong and Zhou (1999), the wealth process admits a unique strong solution such that

$$\mathrm{E}_{t,x,s,z}\left[\sup|X^\pi(\nu)|^\varrho\right]<\infty,$$

for any $\varrho\in[1,+\infty)$ and $(t,x,s,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\{0,1\}$.

Appendix B.
Proof of Theorem 2.7.

Suppose that for $z=1,0$, $V(t,x,s,z)$ and $g(t,x,s,z)$ satisfy verification theorem, and the optimal strategy in HJB equation is achieved at π^*.

Step 1. We aim to show

$$g(t,x,s,z)=\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)],\qquad V(t,x,s,z)=W(t,x,s,z;\pi^*). \tag{25}$$

Due to the condition

$$\mathcal{A}^{\pi^*}g(t,x,s,z)=0,\quad g(T,x,s,z)=x,$$

in the verification theorem, and by Dynkin's formula, we derive

$$\begin{aligned}&\mathrm{E}_{t,x,s,z}[g(T,X^{\pi^*}(T),S(T),z)]\\&=g(t,x,s,z)+\mathrm{E}_{t,x,s,z}\left[\int_t^T\mathcal{A}^{\pi^*}g(\nu,X(\nu),S(\nu),z)\mathrm{d}\nu\right]=g(t,x,s,z),\end{aligned}$$

furthermore, we have

$$g(t,x,s,z)=\mathrm{E}_{t,x,s,z}[g(T,X^{\pi^*}(T),S(T),z)]=\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)],$$

where the variational operator $\mathcal{A}^\pi$ is defined in equation (8) of our paper.

Next we show $V(t,x,s,z)=W(t,x,s,z;\pi^*)$. Since the optimal strategy in HJB equation

$$\sup_{\pi\in\Pi(t,x,s,z)}\left\{\mathcal{A}^\pi V(t,x,s,z)-\mathcal{A}^\pi\frac{\gamma}{2}(g(t,x,s,z))^2+\gamma g(t,x,s,z)\mathcal{A}^\pi g(t,x,s,z)\right\}=0,$$

is achieved at π^*, by the condition

$$\mathcal{A}^{\pi^*}g(t,x,s,z)=0,\quad g(T,x,s,z)=x,$$

in the verification theorem, we rewrite HJB equation as

$$\mathcal{A}^{\pi^*}V(t,x,s,z)-\frac{\gamma}{2}\mathcal{A}^{\pi^*}g^2(t,x,s,z)=0. \tag{26}$$

It follows from

$$V(T,x,s,z)=x,$$

in the verification theorem, Dynkin's formula that

$$\begin{aligned}\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)]&=\mathrm{E}_{t,x,s,z}[V(T,X^{\pi^*}(T),S(T),z)]\\&=V(t,x,s,z)+\int_t^T\mathcal{A}^{\pi^*}V(\nu,X(\nu),S(\nu),z)\mathrm{d}\nu.\end{aligned}$$

Substituting equation (26) into the last equation yields

$$V(t,x,s,z) = \mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)] - \frac{\gamma}{2}\int_t^T \mathcal{A}^{\pi^*} g^2(\nu, X(\nu), S(\nu), z)\mathrm{d}\nu. \tag{27}$$

By Dynkin's formula and boundary conditions, we have

$$\begin{aligned}
&\mathrm{E}_{t,x,s,z}[(X^{\pi^*}(T))^2] = \mathrm{E}_{t,x,s,z}[g^2(T, X^{\pi^*}(T), S(T), z)] \\
&= g^2(t,x,s,z) + \int_t^T \mathcal{A}^{\pi^*} g^2(\nu, X(\nu), S(\nu), z)\mathrm{d}\nu \\
&= \left(\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)]\right)^2 + \int_t^T \mathcal{A}^{\pi^*} g^2(\nu, X(\nu), S(\nu), z)\mathrm{d}\nu,
\end{aligned}$$

or equivalently,

$$\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)] = \int_t^T \mathcal{A}^{\pi^*} g^2(\nu, X(\nu), S(\nu), z)\mathrm{d}\nu. \tag{28}$$

Substituting equation (28) into equation (27) yields

$$V(t,x,s,z) = \mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)] - \frac{\gamma}{2}\mathrm{Var}_{t,x,s,z}[X^{\pi^*}(T)] = W(t,x,s,z;\pi^*).$$

Step 2. We show that π^* is an equilibrium strategy defined in Definition 2.6 in our paper. For any $(\tilde{\pi}_1, \tilde{\pi}_2) \in \mathbb{R}\times\mathbb{R}$, $t \in [0,T]$, and $\varepsilon > 0$, consider the perturbed strategy π_ε defined as

$$\pi_\varepsilon(\nu) = \begin{cases} (\tilde{\pi}_1, \tilde{\pi}_2), & t \le \nu < t+\varepsilon, \\ \pi^*(\nu), & t+\varepsilon \le \nu < T. \end{cases}$$

By

$$W(t,x,s,z;\pi) = \mathrm{E}_{t,x,s,z}[X^{\pi}(T)] - \frac{\gamma}{2}\mathrm{Var}_{t,x,s,z}[X^{\pi}(T)],$$

we derive

$$\begin{aligned}
&\mathrm{E}_{t,x,s,z}[W(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z;\pi_\varepsilon)] - W(t,x,s,z;\pi_\varepsilon) \\
&= \mathrm{E}_{t,x,s,z}\left[\mathrm{E}_{t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z}[X^{\pi^*}(T) - \frac{\gamma}{2}X^{\pi^*}(T)^2] + \frac{\gamma}{2}\left(\mathrm{E}_{t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z}[X^{\pi^*}(T)]\right)^2\right] \\
&-\mathrm{E}_{t,x,s,z}\left[X^{u^*}(T) - \frac{\gamma}{2}X^{\pi^*}(T)^2\right] - \frac{\gamma}{2}\left(\mathrm{E}_{t,x,s,z}[X^{\pi^*}(T)]\right)^2 \\
&= \frac{\gamma}{2}\mathrm{E}_{t,x,s,z}\left[\left(\mathrm{E}_{t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z}[X^{\pi^*}(T)]\right)^2\right] - \frac{\gamma}{2}\left(\mathrm{E}_{t,x,s,z}\left[\mathrm{E}_{t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z}[X^{\pi^*}(T)]\right]\right)^2 \\
&= \frac{\gamma}{2}\mathrm{E}_{t,x,s,z}\left[g^2(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right] - \frac{\gamma}{2}\left(\mathrm{E}_{t,x,s,z}\left[g(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right]\right)^2.
\end{aligned}$$

It follows from the above equation that

$$\begin{aligned}
&W(t,x,s,z;\pi_\varepsilon) \\
&= \mathrm{E}_{t,x,s,z}[W(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z;\pi_\varepsilon)] - \frac{\gamma}{2}\mathrm{E}_{t,x,s,z}\left[g^2(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right] \\
&+\frac{\gamma}{2}\left(\mathrm{E}_{t,x,s,z}\left[g(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right]\right)^2.
\end{aligned}$$

By inserting extra terms $V(t,x,s,z)$ and $g^2(t,x,s,z)$ into the last equation and using the result in step 1, we obtain

$$\begin{aligned}
&W(t,x,s,z;\pi_\varepsilon) = \mathrm{E}_{t,x,s,z}[V(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)] \\
&-\frac{\gamma}{2}\left\{\mathrm{E}_{t,x,s,z}\left[g^2(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right] - g^2(t,x,s,z)\right\} \\
&+\frac{\gamma}{2}\left\{\left(\mathrm{E}_{t,x,s,z}\left[g(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right]\right)^2 - g^2(t,x,s,z)\right\}.
\end{aligned} \tag{29}$$

For any $\pi \in \Pi$, $\varepsilon > 0$ small and $\psi \in C^{1,2}([0,T] \times \mathbb{R} \times \mathbb{R} \times \{0,1\})$, we define the operator

$$\mathcal{A}^{\pi}_{\varepsilon}\psi(t,x,s,z) = \mathrm{E}_{t,x,s,z}[\psi(t+\varepsilon, X^{\pi}(t+\varepsilon), S(t+\varepsilon), z)] - \psi(t,x,s,z), \tag{30}$$

and for small $\varepsilon > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\mathcal{A}^{\pi}_{\varepsilon}\psi(t,x,s,z) = \mathcal{A}^{\pi}\psi(t,x,s,z). \tag{31}$$

By equation (30), we can rewrite equation (29) as

$$\begin{aligned} &W(t,x,s,z;\pi_\varepsilon) = V(t,x,s,z) + \mathcal{A}^{\tilde{\pi}}_{\varepsilon}V(t,x,s,z) - \frac{\gamma}{2}\mathcal{A}^{\tilde{\pi}}_{\varepsilon}g^2(t,x,s,z) \\ &+\frac{\gamma}{2}\left\{\left(\mathrm{E}_{t,x,s,z}\left[g(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)\right]\right)^2 - g^2(t,x,s,z)\right\}. \end{aligned} \tag{32}$$

By Dynkin's formula, we have

$$\mathrm{E}_{t,x,s,z}[g(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z)] = g(t,x,s,z) + \mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right],$$

which implies

$$\begin{aligned} &\left[\mathrm{E}_{t,x,s,z}(g(t+\varepsilon, X^{\tilde{\pi}}(t+\varepsilon), S(t+\varepsilon), z))\right]^2 - g^2(t,x,s,z) \\ &= 2g(t,x,s,z)\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right] + \left\{\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right]\right\}^2. \end{aligned} \tag{33}$$

Substituting equation (33) into equation (32) yields

$$\begin{aligned} &W(t,x,s,z;\pi_\varepsilon) = V(t,x,s,z) + \mathcal{A}^{\tilde{\pi}}_{\varepsilon}V(t,x,s,z) - \frac{\gamma}{2}\mathcal{A}^{\tilde{\pi}}_{\varepsilon}g^2(t,x,s,z) \\ &+\gamma g(t,x,s,z)\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}s\right] + \frac{1}{2}\left\{\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right]\right\}^2. \end{aligned} \tag{34}$$

On the other hand, it follows from the extended HJB equation that

$$\mathcal{A}^{\tilde{\pi}}V(t,x,s,z) - \frac{\gamma}{2}\mathcal{A}^{\tilde{\pi}}g^2(t,x,s,z) + \gamma g(t,x,s,z)\mathcal{A}^{\tilde{\pi}}g(t,x,s,z) \le 0.$$

It follows from equation (31) that for a small enough ε, we have

$$\mathcal{A}^{\tilde{\pi}}_{\varepsilon}V(t,x,s,z) - \frac{\gamma}{2}\mathcal{A}^{\tilde{\pi}}_{\varepsilon}g^2(t,x,s,z) + \gamma g(t,x,s,z)\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right] \le o(\varepsilon). \tag{35}$$

By substituting equation (35) into equation (34), we have

$$\begin{aligned} &W(t,x,s,z;\pi_\varepsilon) \le V(t,x,s,z) + \frac{1}{2}\left\{\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right]\right\}^2 + o(\varepsilon) \\ &= W(t,x,s,z;\pi^*) + \frac{1}{2}\left\{\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right]\right\}^2 + o(\varepsilon). \end{aligned}$$

According to Dominated Convergence Theorem, we have

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0}\left\{\mathrm{E}_{t,x,s,z}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right]\right\}^2 \\ &= \left\{\mathrm{E}_{t,x,s,z}\lim_{\varepsilon \downarrow 0}\left[\int_t^{t+\varepsilon} \mathcal{A}^{\tilde{\pi}}g(\nu, X^{\tilde{\pi}}(\nu), S(\nu), z)\mathrm{d}\nu\right]\right\}^2 = 0, \end{aligned}$$

which implies

$$\liminf_{\varepsilon\downarrow 0}\frac{W(t,x,s,z;\pi^*)-W(t,x,s,z;\pi_\varepsilon)}{\varepsilon}\geq 0.$$

Thus π^* is an equilibrium strategy.

Appendix C.

Proof of Theorem 3.1.

For the post-default case, due to the linear structure of equations (11) and (13) and according to the boundary conditions, we try to guess the solutions in the following form

$$\begin{aligned}
V(t,x,s,1) &= \frac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\frac{A_1(t)}{\gamma}s^{-2\beta}+\frac{B_1(t)}{\gamma}, \quad A_1(T)=B_1(T)=0,\\
g(t,x,s,1) &= \frac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\frac{a_1(t)}{\gamma}s^{-2\beta}+\frac{b_1(t)}{\gamma}, \quad a_1(T)=b_1(T)=0.
\end{aligned} \tag{36}$$

Then we have

$$\begin{aligned}
&V_t(t,x,s,1) = -\frac{1+(\omega-\omega_0-t)r_0}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\frac{A_{1t}(t)}{\gamma}s^{-2\beta}+\frac{B_{1t}(t)}{\gamma},\\
&V_x(t,x,s,1) = \frac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}, \quad V_s(t,x,s,1)=-\frac{2\beta A_1(t)}{\gamma}s^{-2\beta-1},\\
&V_{ss}(t,x,s,1) = \frac{2\beta(2\beta+1)A_1(t)}{\gamma}s^{-2\beta-2}, \quad V_{xx}(t,x,s,1)=V_{xs}(t,x,s,1)=0,\\
&g_t(t,x,s,1) = -\frac{1+(\omega-\omega_0-t)r_0}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}x+\frac{a_{1t}(t)}{\gamma}s^{-2\beta}+\frac{b_{1t}(t)}{\gamma},\\
&g_x(t,x,s,1) = \frac{\omega-\omega_0-t}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}, \quad g_s(t,x,s,1)=-\frac{2\beta a_1(t)}{\gamma}s^{-2\beta-1},\\
&g_{ss}(t,x,s,1) = \frac{2\beta(2\beta+1)a_1(t)}{\gamma}s^{-2\beta-2}, \quad g_{xx}(t,x,s,1)=g_{xs}(t,x,s,1)=0.
\end{aligned} \tag{37}$$

By differentiating equation (9) w.r.t. π, we obtain the optimal investment strategy as follows

$$\begin{aligned}
\pi_1^* &= \frac{(r-r_0)V_x(t,x,s,1)-\gamma\sigma^2 s^{2\beta+1}g_x(t,x,s,1)g_s(t,x,s,1)}{\gamma\sigma^2 s^{2\beta}(g_x(t,x,s,1))^2}\\
&= \frac{(r-r_0)(\omega-\omega_0-T)}{\gamma\sigma^2 s^{2\beta}(\omega-\omega_0-t)}\mathrm{e}^{-r_0(T-t)}+\frac{2\beta(\omega-\omega_0-T)a_1(t)}{\gamma s^{2\beta}(\omega-\omega_0-t)}\mathrm{e}^{-r_0(T-t)}.
\end{aligned} \tag{38}$$

Substituting equations (37) and (38) into equations (11) and (13), we get

$$\begin{aligned}
&\frac{s^{-2\beta}}{\gamma}\left\{A_{1t}(t)-2r\beta A_1(t)+2\beta(r-r_0)a_1(t)+\frac{(r-r_0)^2}{2\sigma^2}\right\}\\
&+\frac{1}{\gamma}\left\{B_{1t}(t)+\beta(2\beta+1)\sigma^2 A_1(t)+\frac{\gamma c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}\right\}=0,\\
&\frac{s^{-2\beta}}{\gamma}\left\{a_{1t}(t)-2r_0\beta a_1(t)+\frac{(r-r_0)^2}{\sigma^2}\right\}\\
&+\frac{1}{\gamma}\left\{b_{1t}(t)+\beta(2\beta+1)\sigma^2 a_1(t)+\frac{\gamma c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}\right\}=0.
\end{aligned} \tag{39}$$

By separating variables, we obtain the following differential equations

$$
\begin{aligned}
&A_{1t}(t) - 2r\beta A_1(t) + 2\beta(r - r_0)a_1(t) + \frac{(r - r_0)^2}{2\sigma^2} = 0,\\
&B_{1t}(t) + \beta(2\beta + 1)\sigma^2 A_1(t) + \frac{\gamma c[\omega - \omega_0 - (1 + a)t]}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)} = 0,\\
&a_{1t}(t) - 2r_0\beta a_1(t) + \frac{(r - r_0)^2}{\sigma^2} = 0,\\
&b_{1t}(t) + \beta(2\beta + 1)\sigma^2 a_1(t) + \frac{\gamma c[\omega - \omega_0 - (1 + a)t]}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)} = 0.
\end{aligned}
\tag{40}
$$

Considering the boundary conditions, we derive

$$
\begin{aligned}
a_1(t) &= \frac{(r - r_0)^2}{2r_0\beta\sigma^2}(1 - \mathrm{e}^{2r_0\beta(t-T)}),\\
b_1(t) &= \frac{(2\beta + 1)(r - r_0)^2}{2r_0}(T - t) - \frac{(2\beta + 1)(r - r_0)^2}{4r_0^2\beta}(1 - \mathrm{e}^{2r_0\beta(t-T)})\\
&+ \frac{\gamma c[r_0(\omega - \omega_0 - (1 + a)t) - (1 + a)]}{r_0^2(\omega - \omega_0 - T)}\mathrm{e}^{r_0(T-t)} - \frac{\gamma c[r_0(\omega - \omega_0 - (1 + a)T) - (1 + a)]}{r_0^2(\omega - \omega_0 - T)},\\
A_1(t) &= \mathrm{e}^{2r\beta t}\int_t^T \mathrm{e}^{-2r\beta\nu}\left[2\beta(r - r_0)a_1(\nu) + \frac{(r - r_0)^2}{2\sigma^2}\right]\mathrm{d}\nu,\\
B_1(t) &= \int_t^T \left[\beta(2\beta + 1)\sigma^2 A_1(\nu) + \frac{\gamma c[\omega - \omega_0 - (1 + a)\nu]}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-\nu)}\right]\mathrm{d}\nu.
\end{aligned}
\tag{41}
$$

Similarly, for the pre-default case, we conjecture the solutions to equations (11) and (14) in the following ways

$$
\begin{aligned}
V(t, x, s, 0) &= \frac{\omega - \omega_0 - t}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)}x + \frac{A_2(t)}{\gamma}s^{-2\beta} + \frac{B_2(t)}{\gamma}, \quad A_2(T) = B_2(T) = 0,\\
g(t, x, s, 0) &= \frac{\omega - \omega_0 - t}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)}x + \frac{a_2(t)}{\gamma}s^{-2\beta} + \frac{b_2(t)}{\gamma}, \quad a_2(T) = b_2(T) = 0.
\end{aligned}
\tag{42}
$$

Then we have

$$
\begin{aligned}
V_t(t, x, s, 0) &= -\frac{1 + (\omega - \omega_0 - t)r_0}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)}x + \frac{A_{2t}(t)}{\gamma}s^{-2\beta} + \frac{B_{2t}(t)}{\gamma},\\
V_x(t, x, s, 0) &= \frac{\omega - \omega_0 - t}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)}, \quad V_s(t, x, s, 0) = -\frac{2\beta A_2(t)}{\gamma}s^{-2\beta-1},\\
V_{ss}(t, x, s, 0) &= \frac{2\beta(2\beta + 1)A_2(t)}{\gamma}s^{-2\beta-2}, \quad V_{xx}(t, x, s, 0) = V_{xs}(t, x, s, 0) = 0,\\
g_t(t, x, s, 0) &= -\frac{1 + (\omega - \omega_0 - t)r_0}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)}x + \frac{a_{2t}(t)}{\gamma}s^{-2\beta} + \frac{b_{2t}(t)}{\gamma},\\
g_x(t, x, s, 0) &= \frac{\omega - \omega_0 - t}{\omega - \omega_0 - T}\mathrm{e}^{r_0(T-t)}, \quad g_s(t, x, s, 0) = -\frac{2\beta a_2(t)}{\gamma}s^{-2\beta-1},\\
g_{ss}(t, x, s, 0) &= \frac{2\beta(2\beta + 1)a_2(t)}{\gamma}s^{-2\beta-2}, \quad g_{xx}(t, x, s, 0) = g_{xs}(t, x, s, 0) = 0.
\end{aligned}
\tag{43}
$$

The first-order of the optimality conditions for problem (13) are given by

$$
\begin{aligned}
\pi_1^* &= \frac{(r - r_0)V_x(t, x, s, 0) - \gamma\sigma^2 s^{2\beta+1}g_x(t, x, s, 0)g_s(t, x, s, 0)}{\gamma\sigma^2 s^{2\beta}(g_x(t, x, s, 0))^2}\\
&= \frac{(r - r_0)(\omega - \omega_0 - T)}{\gamma\sigma^2 s^{2\beta}(\omega - \omega_0 - t)}\mathrm{e}^{-r_0(T-t)} + \frac{2\beta(\omega - \omega_0 - T)a_2(t)}{\gamma s^{2\beta}(\omega - \omega_0 - t)}\mathrm{e}^{-r_0(T-t)},
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\pi_2^* &= \frac{\delta(\omega - \omega_0 - T)}{\gamma h^{\mathrm{P}}\zeta^2(\omega - \omega_0 - t)}\mathrm{e}^{-r_0(T-t)} + \frac{(a_1(t) - a_2(t))(\omega - \omega_0 - T)}{\gamma\zeta s^{2\beta}(\omega - \omega_0 - t)}\mathrm{e}^{-r_0(T-t)}\\
&+ \frac{(b_1(t) - b_2(t) - 1)(\omega - \omega_0 - T)}{\gamma\zeta(\omega - \omega_0 - t)}\mathrm{e}^{-r_0(T-t)}.
\end{aligned}
\tag{45}
$$

Introducing equations (43), (44) and (45) into equations (11) and (14) yields

$$\begin{aligned}&\frac{s^{-2\beta}}{\gamma}\left\{A_{2t}(t)-(2r\beta+h^{\mathrm{P}})A_2(t)+h^{\mathrm{P}}A_1(t)+2\beta(r-r_0)a_2(t)+\left(\frac{\delta}{\zeta}-h^{\mathrm{P}}\right)(a_1(t)-a_2(t))\right.\\&\left.+\frac{(r-r_0)^2}{2\sigma^2}\right\}+\frac{1}{\gamma}\left\{B_{2t}(t)-h^{\mathrm{P}}B_2(t)+h^{\mathrm{P}}B_1(t)+\beta(2\beta+1)\sigma^2A_2(t)+\left(\frac{\delta}{\zeta}-h^{\mathrm{P}}\right)(b_1(t)-b_2(t))\right.\\&\left.+\frac{\gamma c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}+\frac{\delta^2}{2h^{\mathrm{P}}\zeta^2}-\frac{\delta}{\zeta}+\frac{h^{\mathrm{P}}}{2}\right\}=0,\end{aligned}\tag{46}$$

$$\begin{aligned}&\frac{s^{-2\beta}}{\gamma}\left\{a_{2t}(t)-\left(2r_0\beta+\frac{\delta}{\zeta}\right)a_2(t)+\frac{\delta}{\zeta}a_1(t)+\frac{(r-r_0)^2}{\sigma^2}\right\}+\frac{1}{\gamma}\left\{b_{2t}(t)-\frac{\delta}{\zeta}b_2(t)\right.\\&\left.+\frac{\delta}{\zeta}b_1(t)+\beta(2\beta+1)\sigma^2a_2(t)+\frac{\gamma c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}+\frac{\delta^2}{h^{\mathrm{P}}\zeta^2}-\frac{2\delta}{\zeta}+h^{\mathrm{P}}\right\}=0.\end{aligned}\tag{47}$$

By separating variables, we obtain the following differential equations

$$\begin{aligned}&A_{2t}(t)-(2r\beta+h^{\mathrm{P}})A_2(t)+h^{\mathrm{P}}A_1(t)+2\beta(r-r_0)a_2(t)+\left(\frac{\delta}{\zeta}-h^{\mathrm{P}}\right)(a_1(t)-a_2(t))+\frac{(r-r_0)^2}{2\sigma^2}=0,\\&B_{2t}(t)-h^{\mathrm{P}}B_2(t)+h^{\mathrm{P}}B_1(t)+\beta(2\beta+1)\sigma^2A_2(t)+\left(\frac{\delta}{\zeta}-h^{\mathrm{P}}\right)(b_1(t)-b_2(t))\\&\qquad+\frac{\gamma c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}+\frac{\delta^2}{2h^{\mathrm{P}}\zeta^2}-\frac{\delta}{\zeta}+\frac{h^{\mathrm{P}}}{2}=0,\\&a_{2t}(t)-\left(2r_0\beta+\frac{\delta}{\zeta}\right)a_2(t)+\frac{\delta}{\zeta}a_1(t)+\frac{(r-r_0)^2}{\sigma^2}=0,\\&b_{2t}(t)-\frac{\delta}{\zeta}b_2(t)+\frac{\delta}{\zeta}b_1(t)+\beta(2\beta+1)\sigma^2a_2(t)+\frac{\gamma c[\omega-\omega_0-(1+a)t]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-t)}+\frac{\delta^2}{h^{\mathrm{P}}\zeta^2}-\frac{2\delta}{\zeta}+h^{\mathrm{P}}=0.\end{aligned}\tag{48}$$

Taking the boundary conditions into account, we obtain

$$\begin{aligned}a_2(t)&=\frac{(r-r_0)^2}{2r_0\beta\sigma^2}(1-\mathrm{e}^{2r_0\beta(t-T)}),\\b_2(t)&=\frac{(2\beta+1)(r-r_0)^2}{2r_0}(T-t)-\frac{(2\beta+1)(r-r_0)^2}{4r_0^2\beta}(1-\mathrm{e}^{2r_0\beta(t-T)})\\&\quad+\frac{\delta\gamma c[(r_0+\frac{\delta}{\zeta})(r_0(\omega-\omega_0-(1+a)t)-(1+a))-r_0(1+a)]}{\zeta(r_0+\frac{\delta}{\zeta})^2r_0^2(\omega-\omega_0-T)}\mathrm{e}^{r_0(T-t)}\\&\quad-\frac{\delta\gamma c[(r_0+\frac{\delta}{\zeta})(r_0(\omega-\omega_0-(1+a)T)-(1+a))-r_0(1+a)]}{\zeta(r_0+\frac{\delta}{\zeta})^2r_0^2(\omega-\omega_0-T)}\mathrm{e}^{\frac{\delta}{\zeta}(t-T)}\\&\quad-\frac{\gamma c[r_0(\omega-\omega_0-(1+a)T)-(1+a)]}{r_0^2(\omega-\omega_0-T)}(1-\mathrm{e}^{\frac{\delta}{\zeta}(t-T)})\\&\quad+\frac{\gamma c[(r_0+\frac{\delta}{\zeta})(\omega-\omega_0-(1+a)t)-(1+a)]}{(r_0+\frac{\delta}{\zeta})^2(\omega-\omega_0-T)}\mathrm{e}^{r_0(T-t)}\\&\quad-\frac{\gamma c[(r_0+\frac{\delta}{\zeta})(\omega-\omega_0-(1+a)T)-(1+a)]}{(r_0+\frac{\delta}{\zeta})^2(\omega-\omega_0-T)}\mathrm{e}^{\frac{\delta}{\zeta}(t-T)}+\left(\frac{\delta}{h^{\mathrm{P}}\zeta}-2+\frac{h^{\mathrm{P}}\zeta}{\delta}\right)(1-\mathrm{e}^{\frac{\delta}{\zeta}(t-T)}),\end{aligned}\tag{49}$$

$$\begin{aligned}A_2(t)&=\mathrm{e}^{(2r\beta+h^{\mathrm{P}})t}\int_t^T\mathrm{e}^{-(2r\beta+h^{\mathrm{P}})\nu}\left[h^{\mathrm{P}}A_1(\nu)+2\beta(r-r_0)a_2(\nu)+\frac{(r-r_0)^2}{2\sigma^2}\right]\mathrm{d}\nu,\\B_2(t)&=\mathrm{e}^{h^{\mathrm{P}}t}\int_t^T\mathrm{e}^{-h^{\mathrm{P}}\nu}\left[h^{\mathrm{P}}B_1(\nu)+\beta(2\beta+1)\sigma^2A_2(\nu)+\left(\frac{\delta}{\zeta}-h^{\mathrm{P}}\right)(b_1(\nu)-b_2(\nu))\right.\\&\qquad\left.+\frac{\gamma c[\omega-\omega_0-(1+a)\nu]}{\omega-\omega_0-T}\mathrm{e}^{r_0(T-\nu)}+\frac{\delta^2}{2h^{\mathrm{P}}\zeta^2}-\frac{\delta}{\zeta}+\frac{h^{\mathrm{P}}}{2}\right]\mathrm{d}\nu.\end{aligned}\tag{50}$$

Once we get the explicit expressions of $a_1(t)$, $b_1(t)$, $A_1(t)$, $B_1(t)$, $a_2(t)$, $b_2(t)$, $A_2(t)$ and $B_2(t)$, the explicit expressions of the equilibrium investment strategy π_1^*, π_2^* and the equilibrium value function $V(t, x, s, 1)$, $V(t, x, s, 0)$ for the post-default case and the pre-default case can be easily derived. Then the proof of Theorem 3.1 is completed.

References

Barucci, E., Cosso, A., 2015. Portfolio choices and VaR constraint with a defaultable asset. Quantitative Finance 15(5), 853-864.

Bielecki, T.R., Jang, I., 2006. Porfolio optimization with a defaultable security. Asia-Pacific Financial Markets 13, 113-127.

Björk, T., Murgoci, A., 2010. A general theory of Markovian time inconsistent stochastic control problems. Working Paper, Stockholm School of Economics. Available at: `http://ssrn.com/abstract=1694759`.

Björk, T., Murgoci, A., Zhou, X., 2014. Mean variance portfolio optimization with state dependent risk aversion. Mathematical Finance 24, 1-24.

Bo, L., Wang, Y., Yang, X., 2010. An optimal portfolio problem in a defaultable market. Advances in Applied Probability 42, 689-705.

Cairns, A.J.G., Blake, D., Dowd, K., 2006. Stochastic lifestyling: optimal dynamic asset allocation for defined-contribution pension plans. Journal of Economics Dynamics and Control 30, 843-877.

Chen, S., Li, Z., Zeng, Y., 2014. Optimal dividend strategies with time-inconsistent preferences. Journal of Economic Dynamics and Control 46, 150-172.

Capponi, A., Figueroa-López, J.E., 2014. Dynamic portfolio optimization with a defaultable security and regime switching. Mathematical Finance 24, 207-249.

Cox, C.J., Ross, S.A., 1976. The valuation of options for alternative stochastic processes. Journal of Financial Economics 3, 145-166.

Deelstra, G., Grasselli, M., Koehl, P.F., 2004. Optimal design of the guarantee for defined contribution funds. Journal of Economics Dynamics and Control 28, 2239-2260.

Driessen, J., 2005. Is default event risk priced in corporate bonds? Review of Financial Studies 18(1), 165-195.

Duffie, D., Singleton, K.J., 1999. Modeling term structures of defaultable bonds. Review of Financial Studies 12(4), 687-720.

Duffie, D., Singleton, K.J., 2003. Credit Risk: Pricing, Measure, and Management. Princeton University Press, Princeton.

Dufresne, D., 2001. The integrated square root process. Working paper, University of Melbourne.

French, K.R., Schwert, G.W., Stambaugh, R.F., 1987. Expected stock returns and volatility. Journal of Financial Economics 19, 3C29.

Gao, J., 2009. Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model. Insurance: Mathematics and Economics 45, 9-18.

Giacinto, M., Gozzi, F., Federico, S., 2011. Pension funds with a minimum guarantee: a stochastic control approach. Finance and Stochastics. 15, 297-342.

Guan, G., Liang, Z., 2014. Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework. Insurance: Mathematics and Economics 57, 58-66.

Han, N., Hung, M., 2012. Optimal asset allocation for the DC pension plans under inflation. Insurance: Mathematics and Economics 51, 172-181.

He, L., Liang, Z., 2013. Optimal investment strategy for the DC plan with the return of premiums clauses in a mean-variance framework. Insurance: Mathematics and Economics 53, 643-649.

Heston, S.L., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6, 327-343.

Jang, I., 2005. Portfolio optimization with defaultable securities. Thesis, University of New South Wales.

Jarrow, R.A., Turnbull, S.M., 1995. Pricing derivatives on financial securities subject to credit risk. The Journal of Finance 50(1), 53-85.

Jeanblanc, M., Yor, M., Chesney, M., 2009. Mathematical Methods For Financial Markets, Springer-Verlag, London.

Kohler, P.H., Kohler, I., 2000. Frailty modeling for adult and old age mortality: the application of a modified De Moivre Hazard function to sex differentials in mortality. Demogr Res 3, Available at: `http://www.demographic-research.org/Volumes/Vol3/8/`.

Kusuoka, S., 1999. A remark on default risk models. Advances in Mathematical Economics 1, 69-82.

Li, D., Rong, X., Zhao, H., 2015. Time-consistent reinsurance-investment strategy for an insurer and a reinsurer with mean-variance criterion under the CEV model. Journal of Computational and Applied Mathematics 283, 142-162.

Madan, D.P., Unal, H., 1998. Pricing the risks of default. Review of Derivatives Research 2, 121-160.

Pagan, A.R., Schwert, G.W., 1990. Alternative models for conditional stock volatility. Journal of Econometrics 45, 267C290.

Shen, Y., Zeng, Y., 2015. Optimal investment-reinsurance strategy for mean-variance insurers with square-root factor process. Insurance: Mathematics and Economics 62, 118-137.

Stein, E.M., Stein, J.C., 1991. Stock price distribution with stochastic volatility: an analytic approach. Review of Financial Studies 4, 727-752.

Sun, J., Li, Z., Zeng, Y., 2016. Precommitment and equilibrium investment strategies for defined contribution pension plans under a jump-diffusion model. Insurance: Mathematics and Economics 67, 158-172.

Yong, J., Zhou, X., 1999. Stochastic Control: Hamiltonian Systems and HJB Equations. New York: Springer-Verlag, 1999.

Yu, F., 2002. Modeling expected return on defaultable bond. Journal of Fixed Income 12, 69-81.

Zhao, H., Shen, Y., Zeng, Y., 2016. Time-consistent investment-reinsurance strategy for mean-variance insurers with a defaultable security. Journal of Mathematical Analysis and Applications 437, 1036-1057.

Zhu, H., Deng, C., Yue, S., Deng, Y., 2015. Optimal reinsurance and investment problem for an insurer with counterparty risk. Insurance: Mathematics and Economics 61, 242-254.